UNIT-1

MODULE-1

Numerical Analysis - I

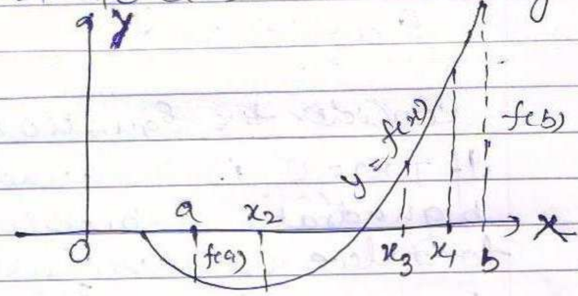
1.1 Solution of Algebric and Transcendental Equation:

Consider the Equation $f(x)=0$
If $f(x)=0$ is a quadratic, Cubic or biquadratic Expression then algebric formulae are available for Expressing the roots. But when $f(x)$ is a Polynomial of higher degree or non Expressing involving transcendental function (Exponential, logrithmic, trignometric) algebric Methods are not available.
Here we shall describle Some numericalmethods for the Solution $f(x)=0$ where $f(x)$ is algebric or transcendental equation or both.

1.2 Bisection (or Bolzano) Method

This method is based on the repeated application of intermediate value property.
Let the function $f(x)$ be Continuous between a and b. For definiteness, Let $f(a)$ be $(-)$ve and $f(b)$ be $(+)$ve.
then first Approximation to the root is
$$x_1 = \frac{1}{2}(a+b).$$

If $f(x_1) = 0$ then x_1 is a root of $f(x)=0$ otherwise root lies b/w a and x_1 or x_1 and b according as $f(x_1)$ is (+)ve or (−)ve. then we bisect this interval as before and continue the process until the root is found to desired accuracy.



In the adjoing figure $f(x_1)$ is (+)ve so that the root lies between 'a' and 'x_1'. Then second approximation to the root is:

$$x_2 = \frac{1}{2}(a + x_1)$$

If $f(x_2)$ is (−ve) then root lies between x_1 and x_2. Then third approximation to the root is:

$$x_3 = \frac{1}{2}(x_1 + x_2)$$ and so on.

Once the method of calculation has been decided. We must describe clearly the computation steps to be followed in a particular sequence. These steps constitute the algorithm of method.

Qu Find the ~~true~~ real root of equation $x\log_{10}x = 1.2$ by Bisection method correct to four decimal places.

Sol: Let $f(x) = x\log_{10}x - 1.2 = 0$

$f(2) = -0.59794$

$f(3) = 0.23136$

Hence a root lies between 2 and 3.

$\therefore$ First approximation to the root is

$$x_1 = \frac{1}{2}(2+3) = 2.5$$

$\therefore \quad f(x_1) = f(2.5) = -0.205 \; (-)ve$

Hence root lies between 2.5 and 3.

Second approximation to the root is.

$$x_2 = \frac{1}{2}[2.5+3)$$

$$x_2 = 2.75$$

Now $f(x_2) = 0.008 \; (+)ve.$

Hence root lies between 2.5 and 2.75.

$\therefore$ Third approximation to the root is

$$x_3 = \frac{1}{2}(2.5 + 2.75)$$

$$x_3 = 2.625$$

Now $f(x_3) = -0.099 \; (-ve).$

Hence root lies beween 2.625 and 2.75.

∴ Fourth approximation to the root is:

$$x_4 = \frac{1}{2}(2.625 + 2.75)$$

$$x_4 = 2.6875$$

Now $f(x_4) = f(2.6875) = -0.046$ (-)ve

Hence root lies between 2.6875 and 2.75.

∴ Fifth approximation:

$$x_5 = \frac{1}{2}(2.6875 + 2.75) = 2.71875$$

$$f(x_5) = f(2.71875) = -0.019 \text{ (-ve)}$$

Hence root lies between 2.71875 and 2.75

∴ Sixth approximation:

$$x_6 = \frac{1}{2}(2.71875 + 2.75)$$

$$x_6 = 2.734375$$

$$f(x_6) = f(2.734375) = -0.06546 \text{ (-)ve}$$

Hence roots lies between 2.73475 and 2.75

∴ Seventh approximation:

$$x_7 = \frac{1}{2}(2.73475 + 2.75)$$

$$x_7 = 2.7421875$$

$$f(x_7) = f(2.7421875) = 0.0013 \text{ (+)ve}$$

Hence root lies between 2.734375 and 2.7421875

∴ Eighth approximation:

$$x_8 = \frac{1}{2}(2.734375 + 2.7421875)$$

$$x_8 = 2.73828125$$

$f(x_8) = f(2.73828125) = -0.002$ (-ve)

Hence root lies between 2.73828125 and 2.7421875.

∴ Nineth approximation:

$$x_9 = \frac{1}{2}(2.73828125 + 2.7421875)$$

$$x_9 = 2.740234$$

$f(x_9) = f(2.740234) = -0.00035$ (-)ve

Hence root lies between 2.7402 and 2.7421

∴ Tenth approximation:

$$x_{10} = \frac{1}{2}(2.7402 + 2.7421)$$

$$x_{10} = 2.74115$$

$f(x_{10}) = f(2.74115) = 0.00043$ (+)ve

Hence the root lies between 2.7402 and 2.74115.

∴ Eleventh approximation:

$$x_{11} = \frac{1}{2}(2.7402 + 2.74115)$$

$$x_{11} = 2.740675$$

$f(x_{11}) = f(2.74675) = 0.000025$ (+)ve

Hence root lies between 2.7402 and 2.740675.

$\therefore$ Twelth approximation:

$$x_{12} = \frac{1}{2}(2.7402 + 2.740675)$$

$$x_{12} = 2.7404$$

$f(x_{12}) = f(2.7404) = -0.00021$ (-ve)

Hence root lies between 2.7404 and 2.70675

$\therefore$ Thirteen approximation:

$$x_{13} = \frac{1}{2}(2.7404 + 2.70675)$$

$$x_{13} = 2.7405$$

$f(x_{13}) = f(2.7405) = -0.00012$ (-ve)

Hence root lies between 2.7405 and 2.740675

$\therefore$ Fourteenth approximation

$$x_{14} = \frac{1}{2}(2.7405 + 2.740675)$$

$$x_{14} = 2.740587$$

$f(x_{14}) = f(2.740587) = -0.00005$ (-ve)

Hence root lies between 2.740587 and 2.740675.

$\therefore$ Fifteenth approximation.

$$x_{15} = \frac{1}{2}(2.740587 + 2.740675)$$

$$x_{15} = 2.7406$$

Since x_{14} and x_{15} are same upto four decimal place.

Hence the approximate real root is 2.7406 Ans

Q. Find root of eqⁿ $x^3 - 2x - 5 = 0$ which lies between 2 and 3 using bisection method is five step.

Dec 2011, Dec 2013.

Here, $f(x) = x^3 - 2x - 5 = 0$

$f(2) = -1$

$f(3) = 16$

Root lies between 2 and 3.

So, here $a = 2$ and $b = 3$.

1st Approximation -

$$x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$f(2.5) = 5.625$ (+ve)

Root lies between 2 and 2.5

2nd Approximation -

$$x_2 = \frac{2+2.5}{2} = 2.25$$

$f(2.25) = 1.890625$ (+ve)

Root lies between 2 and 2.25.

3rd Approximation -

$$x_3 = \frac{2+2.25}{2} = 2.125$$

Date : Page :

$f(2.125) = 0.345$ (+ve)

Root lies between 2 and 2.125

4th Approximation -

$$x_4 = \frac{2 + 2.125}{2} = 2.0625$$

$f(2.0625) = -0.351$ (-ve)

Root lies betwn 2.125 and 2.0625

5th Approximation -

$$x_5 = \frac{2.125 + 2.0625}{2} = 2.09375.$$

Q. Find root of eqn $x^3 - 4x - 9 = 0$ using bi-section method correct to 3 decimal places. NOV-2018.

$f(x) = x^3 - 4x - 9 = 0$

$f(0) = -9$

$f(1) = -12$

$f(2) = -9$ (-ve)

$f(3) = 6$ (+ve)

Root lies between 2 and 3.

So, $a = 2$, $b = 3$.

1st approximation -

$$x_1 = \frac{2+3}{2} = 2.5$$

$f(2.5) = -3.375$ (-ve)

Date : Page :

Root lies between 2.5 and 3.

2nd approximation -

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$f(2.75) = 0.79$ (+ve)

Root lies between 2.5 and 2.75

3rd approximation -

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

$f(2.625) = -1.412$ (-ve)

Root lies between 2.625 and 2.75

4th appriimation -

$$x_4 = \frac{2.75+2.625}{2} = 2.6875$$

$f(2.6875) = -0.339$ (-ve)

Root lies between 2.75 and 2.6875

5th approximation -

$$x_5 = \frac{2.75+2.6875}{2} = 2.7185$$

$f(2.7185) = 0.22$ (+ve)

Date : Page :

Root lies between 2.6875 and 2.7185

6^{th} approximation -

$$x_6 = \frac{2.6875 + 2.7185}{2} = 2.703.$$

$f(2.703) = -0.063.$

Root lies between 2.703 and 2.7185

7^{th} approximation -

$$x_7 = \frac{2.703 + 2.7185}{2} = 2.7094.$$

$f(2.7094) = 0.051 (+ve)$

Root lies between 2.703 and 2.7094.

8^{th} approximation -

$$x_8 = \frac{2.703 + 2.7094}{2} = 2.7062$$

$f(2.7062) = -5.89 \times 10^{-3}$ (-ve)

Root lies between 2.7062 and 2.7094.

9^{th} approximation -

$$x_9 = \frac{2.7062 + 2.7094}{2} = 2.7078.$$

$f(2.7078) = 0.022 (+ve)$

Date : Page :

Root lies between 2.7062 and 2.7078.

10^{th} approximation -

$$x_{10} = \frac{2.7062 + 2.7078}{2} = 2.707$$

$$f(2.707) = 8.48 \times 10^{-3} \ (+ve)$$

Root lies between 2.7062 and 2.707.

11^{th} approximation -

$$x_{11} = \frac{2.7062 + 2.707}{2} = 2.7066$$

1.3 Regula-Falsi Method

OR

Method of False Position

The bisection method guarantees that the iterative process will converge. It is However slow. Thus attempt have been made to speed up bisection method retaining its guaranteed convergence. A method of doing this is called the method of false position.

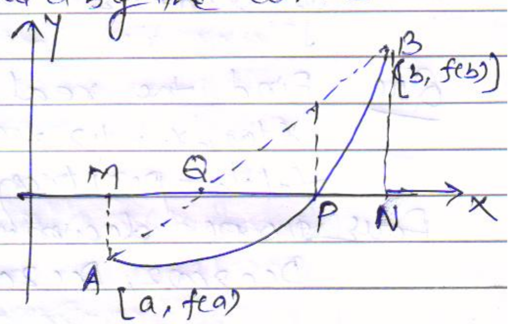
It is some times known as method of linear interpolation.

This is oldest method for finding the real root of numerical equation and closely resembles the bisection method.

In this method.

Let $f(x) = 0$ ——— (1)

Let $y = f(x)$ be represented by the curve AB cuts the x-axis at P.



The real root of (1) is OP.

The false position of curve AB is taken as the chord AB.

The chord AB cuts the x-axis at Q. the Approximate root of $f(x) = 0$ is OQ.

By this method, we find OQ.

Let A [a, f(a)], B [b, f(b)] be the

Extremities of the Chord AB.
the Equation of the Chord AB is

$$y - f(a) = \frac{f(b) - f(a)}{b-a}(x-a)$$

To find OQ, put $y=0$

$$0 - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$(x-a) = -\frac{(b-a)f(a)}{f(b)-f(a)}$$

$$x = a + \frac{(a-b)f(a)}{f(b)-f(a)}$$

$$\boxed{x = \frac{af(b) - bf(a)}{f(b) - f(a)}}$$

We continue the iteration till the root is found to the desired accuracy.
This method is also known as method of linear interpolation.

Que Find the real root of the equation $x\log_{10}x - 1.2 = 0$ by the method of false position method correct to four ~~three~~ decimal places.
Dec 2002, Dec 2005, Dec 2007, June 2014, June 2016.
June - 2009,

Sol Here $f(x) = x\log_{10}x - 1.2 = 0$

$f(2) = 2\log_{10}2 - 1.2 = -0.59794$

$f(3) = 3\log_{10}3 - 1.2 = 0.23136$

One root lies between 2 and 3.

taking $a=2, b=3$

$f(2) = -0.59794, \quad f(3) = 0.23136$

By method of false position, we have

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2(0.23136) - 3(-0.59794)}{0.23136 - (-0.59794)}$$

$x_1 = 2.72102$

$f(2.72102) = 2.72102 \cdot \log_{10} 2.72102 - 1.2$

$= -0.01709$

since $f(2.72102)$ and $f(3)$ are of opposite sign. so the root lies between 2.72102 and 3.

$$x_2 = \frac{2.72102\, f(3) - 3\, f(2.72102)}{f(3) - f(2.72102)}$$

$$= \frac{2.72102(0.23136) - 3(-0.01709)}{0.23136 - (-0.01709)}$$

$x_2 = 2.74021$

Now $f(2.74021) = -0.00038$

16. since $f(2.74021)$ and $f(3)$ are of opposite sign. so the root lies between 2.74021 and 3.

$$x_3 = \frac{2.74021\, f(3) - 3\, f(2.74021)}{f(3) - f(2.74021)}$$

$$x_3 = \frac{2.74021(0.23136) - 3(-0.00038)}{0.23136 - (-0.00038)}$$

$x_3 = 2.74064$

Again $f(x_3) = 2.74064 \log_{10} 2.74064 - 1.2$

$f(3) = -0.00001$

root lies b/w 2.74064 and 3.

$$x_4 = \frac{2.74064 f(3) - 3f(2.74064)}{f(3) - f(2.74064)}$$

$$= \frac{2.74064(0.23136) - 3(-0.00001)}{0.23136 - (-0.00001)}$$

$x_4 = 2.74065$

Hence the root correct to four decimal place is 2.7406

Q Find the root of the Equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by the method of false position.

June-2005, May-2018.

Sol Here we have

$f(x) = x^3 - 5x - 7 = 0$

$f(2) = 8 - 10 - 7 = -9$

$f(3) = 27 - 15 - 7 = +5$

As f(2) and f(3) are of opposite sign. so the root lies between 2 and 3.

First iteration -

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - a f(a)}$$

Here $a = 2$, $b = 3$

$f(2) = -9$ $f(3) = 5$

~~first iteration~~

$$x_1 = \frac{2 f(3) - 3 f(2)}{f(3) - f(2)}$$

$$= \frac{2(5) - 3(-9)}{5 - (-9)} = \frac{37}{14} = 2.6429$$

$x_1 = 2.6429$

Now $f(2.6429) = (2.6429)^3 - 5(2.6429) - 7$

$= -1.7541$

Again f(2.6429) and f(3) are of opposite sign. root lies between 2.6429 and 3.

Second iteration -

$a = 2.6429$, $b = 3$

$$x_2 = \frac{2.6429 f(3) - 3 f(2.6429)}{f(3) - \cancel{(2.6429)} f(2)}$$

$$= \frac{2.6429(5) - 3(-1.7541)}{5 - (-1.7541)}$$

$$= \frac{18.4768}{6.7541} = 2.7356$$

Now $f(2.7356) = (2.7356)^3 - 5(2.7356) - 7$

$= -0.2061$

root lies between ~~2.7356~~ ~~and~~
2.7356 and 3

Third iteration:

$a = 2.7356, \quad b = 3$

$$x_3 = \frac{2.7356\, f(3) - 3\, f(2.7356)}{f(3) - f(2.7356)}$$

$$= \frac{2.7356\,(5) - 3\,(-0.2061)}{5 - (-0.2061)}$$

$$= \frac{14.2963}{5.2061} = 2.7461$$

Now

$$f(2.7461) = (2.7461)^3 - 5(2.7461) - 7$$
$$= -0.02198$$

roots lies between 2.7461 and 3

Four iteration:

$$x_4 = \frac{2.7461\, f(3) - 3\, f(2.7461)}{f(3) - f(2.7461)}$$

$$= \frac{2.7461\,(5) - 3\,(-0.02198)}{5 - (-0.02198)}$$

$$= \frac{13.79644}{5.02198}$$

$$= 2.7472$$

Hence root of given eqn is 2.7472 Ans

Q Find a real root of equation $x^3-2x-5=0$ by method of Regula-Falsi method correct to three decimal place.
June 2010, Dec 2011, Dec 2013.

Sol

Let $f(x) = x^3 - 2x - 5 = 0$

$f(1) = 1 - 2 - 5 = -6$

$f(2) = 8 - 4 - 5 = -1$ (-ve)

$f(3) = 27 - 6 - 5 = 16$ (+ve)

The root lies between 2 and 3.

Let $a = 2$, $b = 3$

$f(2) = -1$ $\quad f(3) = 16$

First iteration -

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{2 f(3) - 3 f(2)}{f(3) - f(2)}$$

$$= \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.0588$$

$x_1 = 2.0588$

Now $f(2.0588) = (2.0588)^3 - 2(2.0588) - 5$

$= -0.3911$ (-ve)

root lies 2.0588 and 3.

Second iteration -

$$x_2 = \frac{2.0588 f(3) - 3 f(2.0588)}{f(3) - f(2.0588)}$$

$$x_2 = \frac{2.0588(16) - 3(-0.3911)}{16 - (-0.3911)}$$

$$= \frac{34.1141}{16.3911} = 2.0813$$

Now

$$f(2.0813) = (2.0813)^3 - 2(2.0813) - 5$$

$$= -0.1468 \ (-ve)$$

root lies b/w 2.0813 and 3

Third iteration

$$x_3 = \frac{2.0813 f(3) - 3f(2.0813)}{f(3) - f(2.0813)}$$

$$= \frac{2.0813(16) - 3(-0.1468)}{16 - (-0.1468)}$$

$$= \frac{33.7412}{16.1468} = 2.0897$$

Now

$$f(2.0897) = (2.0897)^3 - 2(2.0813) - 5$$

$$= -0.0540 \ (-ve)$$

roo lies between 2.0897 and 3

Fourth iteration.

$$x_4 = \frac{2.0897 f(3) - 3f(2.0897)}{f(3) - f(2.0897)}$$

$$= \frac{2.0897(16) - 3(-0.0540)}{16 - (-0.0540)}$$

$$x_4 = \frac{33.5972}{16.0540} = 2.0928$$

Now

$$f(2.0928) = (2.0928)^3 - 2(2.0928) - 5$$
$$= -0.0195$$

root lies between 2.0928 and 3

Fifth iteration

$$x_5 = \frac{2.0928\, f(3) - 3\, f(2.0928)}{f(3) - f(2.0928)}$$

$$x_5 = \frac{2.0928(16) - 3(-0.0195)}{16 - (-0.0195)}$$

$$= \frac{33.5433}{16.0195} = 2.0939$$

Now $f(2.0939) = (2.0939)^3 - 2(2.0939) - 5$
$$= -0.0073$$

root lies between 2.0939 and 3

Sixth iteration -

$$x_6 = \frac{2.0939\, f(3) - 3\, f(2.0939)}{f(3) - f(2.0939)}$$

$$= \frac{2.0939(16) - 3(-0.0073)}{16 - (-0.0073)}$$

$$= \frac{33.5243}{16.0073} = 2.0943$$

Now $f(2.0943) = (2.0943)^3 - 2(2.0943) - 5$
$$= -0.0028$$

Seventh iteration

$$x_7 = \frac{2.0943\, f(3) - 3\, f(2.0943)}{f(3) - f(2.0943)}$$

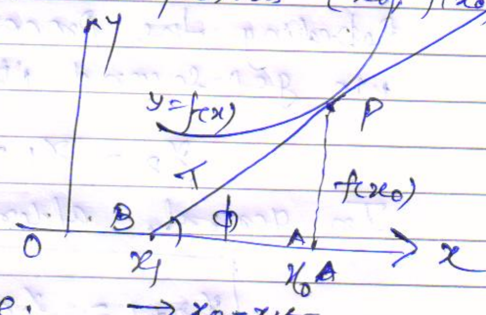
$$x_7 = \frac{2.0943\,(16) - 3\,(-0.0028)}{16 - (-0.0028)}$$

$$= \frac{33.5172}{16.0028} = 2.0945$$

Hence $x_6 = x_7$ are correct to three decimal place so required root of the given equation is 2.094

1.4 Newton-Raphson's Method:-

The solution to an equation $f(x)=0$ may oftain be found by a simple procedure called Newton-Raphson method.
Let $x=x_0$ be the initial or starting value of the root. This method is generally used to improve the result obtained by the previous method.
This method consists of relacing the part of the curve between points $[x_0, f(x_0)]$ and the x-axis by means of the tangent to the curve at the point and is described graphically in adjoining figure.



The intercept OT on the x-axis of the tangent to the curve at the point P. is taken as the first approximation

from the figure; $\tan\phi = \dfrac{AP}{BA}$

$$\tan\phi = \frac{f(x_0)}{x_0 - x_1}$$

But $\tan\phi = \dfrac{dy}{dx} = f'(x_0)$ $\because y = f(x)$

this gives

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$(x_0 - x_1) f'(x_0) = f(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Repeating the process replacing x_0 by x_1 we get second iteration as.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$ and so on.

In general, after $(n+1)$ iteration, we get

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

where $n = 0, 1, 2, 3 \ldots$

Q. Find a real root of equation $x^4 - x - 10 = 0$ correct to three decimal place by using Newton Raphson method. Dec 2003, Dec 2011

$f(x) = x^4 - x - 10 = 0$. $\quad f(0) = -10$

$$\left.\begin{aligned} f(1) &= -10 \,(-ve) \\ f(2) &= 4 \,(+ve) \end{aligned}\right\} \text{root lies b/w 1 \& 2}$$

$$x_0 = \frac{1+2}{2} = 1.5$$

1st Approx

$(n=0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \left[\frac{(1.5)^4 - 1.5 - 10}{4(1.5)^3 - 1}\right] = 2.015$$

NOTE : Continue finding root untila value upto 3 decimal will be equal.

2^{nd} approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.015 - \left[\frac{(2.015)^4 - 2.015 - 10}{4(2.015)^3 - 1}\right]$$

$$x_2 = 1.87409.$$

3^{rd} approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.87409 - \left[\frac{(1.87409)^4 - 1.87409 - 10}{4(1.87409)^3 - 1}\right]$$

$$x_3 = 1.85586.$$

4^{th} approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 1.85586 - \left[\frac{(1.85586)^4 - 1.85586 - 10}{4(1.85586)^3 - 1}\right]$$

$$x_4 = 1.85586$$

3^{rd} and 4^{th} approximation are same.
Hence, one root is 1.8558.

Q. Find a root by N-R method of eq^n $x^3 - 3x + 1 = 0$ correct to 3 decimal place.

Dec 2001, Dec 2007,
June 2017

Date : Page :

$f(x) = x^3 - 3x + 1$

$x = 0 \quad f(0) = 1$

$x = 1 \quad f(1) = -1.$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} x_0 = \frac{0+1}{2} = 0.5.$

1st approximation, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_1 = 0.5 - \frac{(0.5)^3 + 3 \times 0.5 - 1}{3(0.5)^2 - 3}.$$

$x_1 = 0.3333.$

2nd approximation, $x_2 = 0.3333 - \frac{(0.3333)^3 + 3 \times 0.3333 - 1}{3(0.3333)^2 - 3}.$

$x_2 = 0.34721$

3rd approximation, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.34721 - \frac{(0.34721)^3 + 3 \times 0.34721 - 1}{3(0.34721)^2 - 3}$$

$x_3 = 0.34729$

2nd and 3rd approximation is same.

Hence, one root is 0.3472.

Q. Apply Newton Raphson method to solve $3x - \cos x - 1 = 0$ correct to three decimal place.

Dec 2002, June 2004, Dec 2006, June 2011, June 2016, May-2018, Nov-2018

$f(x) = 3x - \cos x - 1 = 0.$

$f(0) = -2 \quad (-ve)$

$f(1) = 1.459 \quad (+ve)$

$\Rightarrow x_0 = 0.5$

Date : Page :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{3(0.5) - \cos(0.5) - 1}{3 + \sin(0.5)}$$

$$x_1 = 0.6085$$

$$x_2 = 0.6085 - \frac{3(0.6085) - \cos(0.6085) - 1}{3 + \sin(0.6085)}$$

$$x_2 = 0.6071$$

$$x_3 = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)}$$

$$x_3 = 0.6071$$

2^{nd} and 3^{rd} approximation is same.
Hence, one root is 0.6071.

Q. Using Newton Raphson method find real root of $x \log_{10} x - 1.2 = 0$ Correct to five decimal places.
Feb. 2010

$$f(x) = x \log_{10} x - 1.2 = 0$$
$$= x(0.4343) \log_e x - 1.2$$

$f(2) = -0.597940$ (-ve)
$f(3) = 0.2313$ (+ve)

Date :   Page :

Root lies between 2 and 3.

$$f(x) = x \log_{10} x - 1.2 = 0$$
$$= x\left(\frac{\log_e 10}{\log e}\right) - 1.2$$

$$f(x) = x\,[0.4343 \log_e x] - 1.2$$
$$f'(x) = 0.4343\left[x \cdot \frac{1}{x} + \log_e x\right] - 0$$

$$f'x = 0.4343 + 0.4343 \log_e x.$$
$$= 0.4343\,(1 + \log_e x)$$

$$x_0 = \frac{2+3}{2} = 2.5$$

1^{st} approximation, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 2.5 - \frac{[(2.5)\log_{10}(2.5) - 1.2]}{[0.4343(1 + \log_e 2.5)]}$$

$$x_1 = 2.74650$$

2^{nd} approximation, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 2.74650 - \left(\frac{(274650)\log_{10} 2.74650 - 1.2}{0.4343(1 + \log_e 2.74650)}\right).$$

$$= 2.740649.$$

3^{rd} approximation, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

Date : Page :

$$= 2.740649 - \left(\frac{(2.740649) \log_{10} (2.70649) - 1.2}{0.4343 (1 + \log e^{2.70649})} \right)$$

2) $x_3 = 2.74064.$

Root is 2.74064.

Q. Find a root of eq^n $xe^x - \cos x$ by N-R method upto three decimal place.

$f(x) = x \cdot e^x - \cos x$

$f(0) = 0.1 = -1$

$f(1) = 1.77$

$x_0 = \frac{0+1}{2} = 0.5$

$f'(x) = x \cdot e^x + e^x \cdot 1 - \sin x$

$= (x+1) e^x + \sin x.$

$x_1 = x_n - \frac{f(x_n)}{f'(x_n)}$

$n = 0.$

$x_1 = 0.5 - \frac{(0.5) e^{0.5} - \cos 0.5}{(0.5+1) e^{0.5} + \sin 0.5}$

$= 0.57075537$

Q. Evaluate $\sqrt{12}$ to four decimal place by Newton - Raphson method. Dec 2012

Date : Page :

Let $x = \sqrt{12}$

$x^2 = 12$

$f(x) = x^2 - 12 = 0$

$f(3) = -3$

$f(4) = 4$

Root lies between +3 and 4

$x_0 = \frac{3+4}{2} = 3.5$

I^{st} approximation, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 3.5 - \left(\frac{(3.5)^2 - 12}{2(3.5)}\right)$

$x_1 = 3.46429.$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 3.46429 - \left(\frac{(3.46429)^2 - 12}{2(3.46429)}\right)$

$= 3.46410.$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 3.46410 - \left(\frac{(3.46410)^2 - 12}{2(3.46410)}\right)$

$= 3.4640$

Date : Page :

Q. Find the negative root of eq^n $x^3 - 21x + 3500 = 0$ correct to three decimal place by newton - raphson method.

Dec-2013

$f(x) = x^3 - 21x + 3500$

$x' = 3x^2 - 21$

$f(-15) = 440$

$f(-16) = -260$

Root lies betw^n -15 & -16.

$x_0 = \frac{-15-16}{2} = -15.5$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= -15.5 - \left[\frac{(-15.5)^3 - 21(-15.5) + 3500}{3(-15.5)^2 - 21}\right]$

$= -15.6452$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= -15.6452 - \left[\frac{(-15.6452)^3 - 21(-15.6452) + 3500}{3(-15.6452)^2 - 21}\right]$

$= -15.6438$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= -15.6438 - \left[\frac{(-15.6438)^3 - 21(-15.6438) + 3500}{3(-15.6438)^2 - 21}\right]$

$= -15.6438$

Date : Page :

* OPERATORS

Different types of operators

(i) The Shifting operator (E)

The shift operator denoted by 'E'. It is operation of increasing the argument 'x' by 'h'.

i.e., $E f(x) = f(x+h)$

$$E^2 f(x) = f(x+2h)$$

$$\vdots \qquad \vdots$$

$$E^n f(x) = f(x+nh)$$

(ii) Forward difference operator (Δ)

It is denoted by 'Δ' and defined by

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

OR

$$\boxed{\Delta y_0 = y_1 - y_0}$$

(iii) Backward difference operator (∇)

It is denoted by '∇' and defined by

$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

OR

$$\boxed{\nabla y_i = y_i - y_{i-1}} \qquad \text{where } i = 1, 2, 3 \ldots$$

(iv) Central difference operator (δ)

$$\boxed{\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}$$

Date : Page :

(v) Avarage operator (μ)
It is denoted by 'μ' and defined by

$$\boxed{\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]}$$

(vi) Differential operator (D)
It is denoted by 'D' and defined by

$$\boxed{D f(x) = \frac{d}{dx} f(x)}$$

* RELATION between OPERATORS :-

(i) $\Delta = E - 1$ or $E = 1 - \Delta$

we have,

$$\Delta f(x) = f(x+h) - f(x)$$
$$= E f(x) - f(x)$$
$$\Delta f(x) = (E-1) f(x)$$
$$\boxed{\Delta = E - 1}$$ (by right cancellation law)

(ii) $\nabla = 1 - E^{-1}$

we have,

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x)$$
$$= (1-E^{-1}) f(x)$$
$$\boxed{\nabla = 1 - E^{-1}}$$

Date : Page :

(iii) $\delta = \left[E^{1/2} - E^{-1/2} \right]$.

We have,

$\delta f(x) = f(x+h/2) - f(x-h/2)$

$= E^{1/2} f(x) - E^{-1/2} f(x)$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

(iv) $\mu = \frac{1}{2} \left[E^{1/2} + E^{-1/2} \right]$.

We have,

$\mu f(x) = \frac{1}{2} \left[f(x+h/2) + f(x-h/2) \right]$

$= \frac{1}{2} \left[E^{1/2} f(x) + E^{-1/2} f(x) \right]$

$= \frac{1}{2} \left[E^{1/2} + E^{-1/2} \right] f(x)$

$$\boxed{\mu = \frac{1}{2} \left[E^{1/2} + E^{-1/2} \right]}$$

Q. Prove that $E = e^{hD}$ June 2011

We have $Ef(x) = f(x+h)$

By Taylor theorem,

$Ef(x) = f(x) + h \cdot f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \cdots$

$= f(x) + h \cdot Df(x) + \frac{h^2}{\lfloor 2} D^2 f(x) + \cdots$

$= \left[1 + (hD) + \frac{(hD)^2}{\lfloor 2} + \frac{(hD)^3}{\lfloor 3} + \cdots \right] f(x)$

Date : Page :

$E f(x) = e^{hD} f(x)$

$$\boxed{E = e^{hD}}$$

Q. Prove that $\log(1+\Delta) = -\log(1-\nabla) = hD$
Dec-2005, Dec-2011

We have

$E = e^{hD}$

$hD = \log E$

$hD = \log(1+\Delta)$

$\therefore E = 1+\Delta$.

We have,

$\nabla = 1 - E^{-1}$

$\nabla = 1 - (e^{hD})^{-1}$

$\nabla - 1 = -(e^{hD})^{-1}$

$1 - \nabla = (e^{hD})^{-1}$

$(1-\nabla) = e^{-hD}$

$-hD = \log(1-\nabla)$

$$\boxed{hD = -\log(1-\nabla)}$$

Q. Prove that $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (2+\Delta)$
Dec-2005

$(E^{1/2} + E^{-1/2})(E^{1/2})$

$= E + E^0$

$= E + 1$

$= (1+\Delta) + 1$

$= 2 + \Delta$

Q. Evaluate

$(1+\Delta)(1-\nabla)$

$(\cancel{1}+E-\cancel{1})(1-(1-E^{-1}))$

$E[\cancel{1}-\cancel{1}+E^{-1}]$

$E[E^{-1}]$

$= 1.$

Q. Evaluate $(\nabla+\Delta)^2(x^2+x)$

Taking $h = 1$

$((1-E^{-1})+(E-1))^2(x^2+x)$

$(E-E^{-1})^2(x^2+x)$

$(E^2+E^{-2}-2)(x^2+x)$

$E^2x^1+E^{-2}x^2-2x^2+E^2x+E^{-2}x-2x$

$((x+2h)^2+(x-2h)^2-2x^2)+((x+2h)+(x-2h)-2x)$

$((x+2)^2+(x-2)^2-2x^2)+((x+2)+(x-2)-2x)$

$(x+2)^2+(x-2)^2-2x^2+(x+2)+(x-2)-2x$

$\cancel{x^2}+4+\cancel{4x}+\cancel{x^2}+4-\cancel{4x}-\cancel{2x^2}+\cancel{x}+\cancel{2}+\cancel{x}-\cancel{2}-\cancel{2x}$

$= 8$

Q. Prove that

$\nabla\cdot\Delta = \Delta-\nabla = \delta^2$

$\nabla\cdot\Delta = (1-E^{-1})(E-1)$

$= (E+E^{-1}-2)$

$= (E^{1/2}-E^{-1/2})^2$

$\boxed{\nabla\Delta = \delta^2}$

Date : Page :

$$\Delta - \nabla = (E-1) - (1-E^{-1})$$
$$= E + E^{-1} - 2$$
$$= (E^{1/2} + E^{-1/2})$$
$$\boxed{\Delta - \nabla = \delta^2}$$

Q. Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$$RHS = \frac{\Delta^2 - \nabla^2}{\nabla\Delta}$$
$$= \frac{(\Delta+\nabla)(\Delta-\nabla)}{\nabla\Delta}$$
$$= \frac{(\Delta+\nabla)\cancel{\delta^2}}{\cancel{\delta^2}}$$
$$= \Delta + \nabla = LHS.$$

Q. Prove that

$$\nabla = \delta E^{1/2}$$
$$\nabla = 1 - E^{-1}$$

$$RHS = \delta E^{1/2}$$
$$= (E^{1/2} + E^{-1/2})\, E^{1/2}$$
$$= E + 1$$
$$= 1 - E^{-1} = \nabla \text{ proved}$$

Q. Prove that

$$\delta = \Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2}$$

$$\Delta(1+\Delta)^{-1/2} = E^{-1}(1+(E-1))^{-1/2}$$
$$= E^{-1}(\cancel{1}+E-\cancel{1})^{-1/2}$$
$$= E^{-1}(E^{-1/2})$$
$$= E^{1/2} - E^{-1/2} = \delta$$

$$\nabla(1-\nabla)^{-1/2} = 1-E^{-1}(1-(1-E^{-1}))^{-1/2}$$
$$= 1-E^{-1}(1-1+E^{-1})^{-1/2}$$
$$= 1-E^{-1}(E^{1/2})$$
$$= E^{1/2} - E^{-1/2} = \delta$$

Q. Prove that

$$e^x = \left(\frac{\Delta^2}{E}\right) e^x \; \frac{Ee^x}{\Delta^2 e^x}$$

$$Ee^x = e^{x+h}$$
$$\Delta^2 e^x = (E-1)^2 e^x$$
$$= (E^2+1-2E)\, e^x$$
$$= (e^{x+2h} + e^x - 2e^{x+h})$$
$$= e^x(e^{2h}+1-2e^h)$$

$$\Delta^2 e^x = e^x(e^h-1)^2$$

Now RHS $= \left(\frac{\Delta^2 e^x}{E}\right)\left(\frac{Ee^x}{\Delta^2 e^x}\right)$

$$= \left(\frac{e^x \cdot (e^h-1)^2}{E}\right)\left(\frac{e^{x+h}}{e^x(e^h-1)^2}\right)$$
$$= E^{-1}\cancel{(e^x)}\cancel{(e^h-1)^2} \cdot \frac{e^{x+h}}{\cancel{e^x(e^h-1)^2}}$$
$$= E^{-1}(e^x) \cdot e^{\cancel{x}+h-\cancel{x}}$$
$$= e^{x-h} \cdot e^h = e^{x-\cancel{h}+\cancel{h}} = e^x$$

* Factorial Notation

Consider the continued product $x(x-h)(x-2h)\cdots\cdots(x-(n-1)h)$ containing n factors s of which x is the first one and the successive difference h. This is known as factorial polynomials and It is denoted by $x^{(n)}$.

i.e., $$\boxed{x^{(n)} = x(x-h)(x-2h)\cdots\cdots(x-(n-1)h)}$$

Particular case.

If $h=1$

then
$$x^{(1)} = x$$
$$x^{(2)} = x(x-1)$$
$$x^{(3)} = x(x-1)(x-2)$$
$$\vdots \qquad \vdots$$
$$x^{(n)} = \frac{x(x-1)(x-2)\cdots\cdots(x(n-1))\lfloor x-n}{\lfloor x-n}$$

$$\boxed{x^{(n)} = \frac{\lfloor x}{\lfloor x-n}}$$

• Differences of a Factorial Polynomials.

$$\Delta x^{(n)} = (E-1)(x^{(n)})$$
$$= E x^{(n)} - x^{(n)}$$
$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= \{(x+h)((x+h)-h)((x+h)-2h) \text{---} ((x+h)-nh)\}$$
$$- \{x(x-h)(x-2h) \text{---} (x-(n-1)h\}$$

$$\Delta x^{(n)} = \{x(x+h)(x-h) \text{---} (x-(n-2)h\}$$
$$- \{x(x-h)(x-2h) \text{----} (x-(n-1)h\}$$

$$\Delta x^{(n)} = x(x-h)(x-2h) \text{---} (x-(n-2)h)$$
$$\{(x+h) - \{x-(n-1)h\}$$

$$= x(x-h)(x-2h) \text{---} (x-(n-2)h)$$
$$(\cancel{x}+\cancel{h}-\cancel{x}+nh-\cancel{h})$$
$$= nhx(x-h)(x-2h) \text{---} (x-(n-2)h)$$

$$\boxed{\Delta x^{(n)} = nh\, x^{(n-1)}}$$

Similarly,

$$\Delta^2 x^{(n)} = n(n-1)\, x^{(n-2)}\, h^2$$
$$\Delta^3 x^{(n)} = n(n-1)(n-2).\, x^{(n-3)}\, h^3.$$
$$\vdots$$
$$\Delta^n x^{(n)} = n(n-1)(n-2) \text{---} 3.2.1\, h^n.$$

$$\boxed{\Delta^n x^{(n)} = \lfloor n\, h^n.}$$

Date : Page :

• Polynomial in Factorial Notation

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
be the polynomial of n^{th} degree to be expressed in factorial notation
$x^{(n)}, \; x^{(n-1)}, \; x^{(n-2)} \; \ldots$ etc
are respectively polynomials of n^{th} degree. $(n-1)^{th}$ degree, $(n-2)^{th}$ degree .. etc.
We can express $f(x)$ as -

~~$f(x) = A_0 + A_1(x)$~~

$$f(x) = A_0 + A_1 x^{(1)} + A_2 x^{(2)} + \cdots + A_n x^{(n)} \quad \text{--- (1)}.$$

where $A_0, A_1, A_2 \ldots A_n$ are to be determined.

$$\text{Now, } \Delta f(x) = A_1 + 2A_2 x^{(1)} + 3A_3 x^{(2)} + \cdots + nA_n x^{(n-1)}$$

$$\Delta^2 f(x) = 2A_2 + 6A_3 x^{(1)} + \cdots + n(n-1)A_n x^{(n-2)}$$

$$\Delta^3 f(x) = 6A_3 + \cdots + n(n-1)(n-2)A_n x^{(n-3)}$$

$$\vdots \qquad\qquad \vdots$$

$$\Delta^n f(x) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 \, A_n x^{(0)}$$

$$\boxed{\Delta^n f(x) = \underline{|n} \cdot A_n}$$

Putting $x = 0$.

$$\boxed{f(0) = A_0}$$

$$\boxed{\Delta f(0) = A_1}$$

Date : Page :

$\Delta^2 f(0) = 2A_2$ $\boxed{A_2 = \frac{1}{2}\Delta^2 f(0)}$

$\Delta^3 f(0) = 6A_3$. $\boxed{A_3 = \frac{1}{\lfloor 3}\Delta^3 f(0)}$

$\vdots$ $\vdots$

$\Delta^n f(0) = A_n \lfloor n \Rightarrow \boxed{A_n = \frac{\Delta^n f(0)}{\lfloor n}}$

Hence,

$$f(x) = f(0) + \Delta f(0)\, x^{(1)} + \frac{\Delta^2 f(0)}{\lfloor 2} x^{(2)} + \frac{\Delta^3 f(0)}{\lfloor 3} x^{(3)} + \cdots\cdots + \frac{\Delta^n f(0)}{\lfloor n} x^{(n)}.$$

REMARK :-

Any polynomial of degree 'n' can be expressed as a factorial polynomial of the same degree and vice - versa.

Q. Represent the function $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$. and its successive difference in factorial notation in which interval of differencing is one.

<u>1st Method</u> (Forward Method)

The value of $f(x)$ at $x = 0, 1, 2, 3, 4.$
are $9, 10, 37, 54, 49.$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	$9 = f(0)$				
		$1 = \Delta f(0)$			
1	10		$26 = \Delta^2 f(0)$		
		27		$-36 = \Delta^3 f(0)$	
2	37		-10		$24 = \Delta^4 f(0)$
		17		-12	
3	54		-22		
		-5			
4	49.				

In factorial notation,

$$f(x) = f(0) + \frac{\Delta f(0)}{\lfloor 1} x^{(1)} + \frac{\Delta^2 f(0)}{\lfloor 2} x^{(2)} + \frac{\Delta^3 f(0)}{\lfloor 3} x^{(3)}$$

$$+ \frac{\Delta^4 f(0)}{\lfloor 4} x^{(4)}$$

$$f(x) = 9 + x + \frac{26 x^2}{2} - \frac{36 x^3}{6} + \frac{24 x^4}{24}$$

$$f(x) = 9 + \underbrace{x^{(1)}}_{x} + 13 \underbrace{x^{(2)}}_{x(x-1)} - 6 \underbrace{x^{(3)}}_{x(x-1)(x-2)} + \underbrace{x^{(4)}}_{x(x-1)(x-2)(x-3)}$$ Ans

Date : Page :

$\Delta f(x) = 1 + 26x^{(1)} - 18x^{(2)} + 4x^{(3)}$

$\Delta^2 f(x) = 26 - 36x + 12x^2$

$\Delta^3 f(x) = -36x + 24x$

$\boxed{\Delta^4 f(x) = 24.}$

2nd Method (Direct Method)

Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$

$= ax^{(4)} + bx^{(3)} + cx^{(2)} + dx^{(1)} + e.$

be the factorial notation

$x^4 - 12x^3 + 42x^2 - 30x + 9$

$= ax(x-1)(x-2)(x-3) + bx(x-1)(x-2) + cx(x-1) + dx + e$

Put $x = 0 \Rightarrow \boxed{e = 9}$

$x = 1$ $\boxed{d = 1}$

$x = 2$ we get $2c + 2d + e = 37$

$\boxed{c = 13.}$

$\boxed{b = -6.}$ $\boxed{a = 1}$

Hence,

$f(x) = x^{(4)} - 6x^{(3)} - 13x^{(2)} + x^{(1)} + 9$

be the factorial notation.

$dx = -30x$

$d = 1$

$ax^{(4)}$ $dx + e = 9$

$dx = -9$

$d =$

$d =$

$-30x + 9 = dx + e$

$-30x + 9$

$-21 = d + 9$

$d =$

Date : Page :

3rd Method (Synthetic Division Method).

Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$

$$= Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$$

be the factorial polynomial.

It is divided by polynomial

x, $(x-1)$, $(x-2)$, and $(x-3)$

we have,

	1	-12	42	-30	9
0 multiply	0 equal →	0 ↓add.	0	0	0
1	1	-12	42	-30	$\boxed{9 = E}$
		1	-11	31	
2	1	-11	31	$\boxed{1 = D}$	
		2	-18		
3.	1.	-9.	$\boxed{13 = C}$		
		3			
	$\boxed{1 = A}$	$\boxed{-6 = B}$			

$$f(x) = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9.$$

Date : Page :

Q. Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation & hence show that $\Delta^3 y = 12$.

$$f(x) = 2x^3 - 3x^2 + 3x - 10$$
$$= Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$

be the factorial polynomial

It is divided by polynomial

$x, (x-1), (x-2),$

we have

0	2	-3	3	-10
	0	0	0	0
1	2	-3	3	$\boxed{-10} = D$
		2	-1	2
2	2	-1	$\boxed{2} = C$	-8
		$\boxed{4}$		
	$\boxed{2} = A$	$\boxed{+3} = B$		

$$\therefore \quad y = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10$$
$$\Delta y = 6x^{(2)} + 6x^{(1)} + 2$$
$$\Delta^2 y = 12x^{(1)} + 6$$
$$\boxed{\Delta^3 y = 12} \text{ Ans}$$

Date : Page :

Q. Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$

We are to given

$$\Delta f(x) = 2x^3 + 3x^2 - 5x + 4$$

$$= \cancel{A(x)}$$

$$= Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$

be factorial polynomial.

$$\Delta f(x) = 2x^{(3)} + 9x^{(2)} + 0x^{(1)} + 4. \quad \text{(After applying synthetic division)}$$

0	2	3	-5	4.
1	2	3	-5	$\boxed{4}$ = D.
		2	5	
2	2	5	0 = C.	
		4		
3.	$\boxed{2}$ = A.	$\boxed{9}$ = B		

$$\Delta f(x) = 2x^{(3)} + 9x^{(2)} + 0x^{(1)} + 4.$$

$$f(x) = \frac{\cancel{2}x^{(4)}}{\cancel{4}2} + \frac{\overset{3}{\cancel{9}}x^{(3)}}{\cancel{3}} + 4x^{(1)}$$

$$f(x) = \frac{x^{(4)}}{2} + 3x^{(3)} + 4x^{(1)}$$

$$f(x) = \frac{x(x-1)(x-2)(x-3)}{2} + 3x(x-1)(x-2) + 4x.$$

$$= \frac{1}{2}(x^2-x)(x-2)(x-3) + 3(x^2-x)(x-2) + 4x$$

$$= \frac{1}{2}(x^3-2x^2-x^2+2x)(x-3) + 3x^3-6x^2-3x^2+6x+4x$$

Date : Page :

Q. Obtain the function whose first difference $9x^2 + 11x + 5$.

$9x^2 + 11x + 5$

$Ax^{(2)} + Bx^{(1)} + C$

be factorial polynomial

0	9	11	5
1	9	11	$\boxed{5} = C$
		9	$\cancel{29}$
	$\boxed{9} = A$	$\boxed{20} = B$	$\cancel{25}$

$\Delta f(x) = 9x^{(2)} + 20x^{(1)} + 5$

$f(x) = \frac{3\cancel{9}x^{(3)}}{\cancel{3}} + \frac{\overset{10}{\cancel{20}}x^{(2)}}{\cancel{2}} + 5x^{(1)}$

$f(x) = 3x(x-1)(x-2) + 10x(x-1) + 5x$

$= 3(x^2 - x)(x-2) + 10x^2 - 10x + 5x$

$= 3(x^3 - 2x^2 - x^2 + 2x) + 10x^2 - 10x + 5x$

$= 3x^3 - \underline{6x^2} - \underline{3x^2} + 6x + \underline{10x^2} - 10x + 5x$

$f(x) = 3x^3 + x^2 + x$

Date :   Page :

Q. Obtain the function whose first difference is $x^3 + 4x^2 + 9x + 12$

Sol. Let $f(x) = Ax^{(3)} + Bx^{(2)} + 9x^{(1)} + C$ in factorial notation

0	1	4	9	12
		0	0	0
1	1	4	9	$\boxed{12} = D$
		1	5	
2	1	5	$\boxed{13} = C$	
		2		
3	$\boxed{1} = A$	$\boxed{7} = B$		

$$\Delta f(x) = x^{(3)} + 7x^{(2)} + 13x^{(1)} + 12$$

$$f(x) = \frac{x^4}{4} + \frac{7x^3}{3} + \frac{13x^2}{2} + 12x$$

$$= \frac{1}{4} x(x-1)(x-2)(x-3) + \frac{7}{3} x(x-1)(x-2) + \frac{13}{2} x(x-1) + 12x$$

$$f(x) = \frac{1}{4}(x^2-x)(x-2)(x-3) + \frac{7}{3}(x^2-x)(x-2) + \frac{13}{2}(x^2-x) + 12x$$

$$= \frac{1}{4}(x^3 - 2x^2 - x^2 + 2x)(x-3) + \frac{7}{3}(x^3 - 2x^2 - x^2 + 2x) + \frac{13x^2}{2} - \frac{13x}{2} + 12x$$

$$= \frac{x^4}{4} - \frac{3x^3}{4} - \frac{2x^3}{4} + \frac{6x^2}{4} - \frac{x^3}{4} + \frac{3x^2}{4} + \frac{2x^3}{4} - \frac{6x}{4} + \frac{7x^3}{3} - \frac{14x^2}{3} - \frac{7x^2}{3} + \frac{14x}{3} + \frac{13x^2}{2} - \frac{13x}{2} + 12x$$

$$= \frac{x^4}{4} + \left(\frac{-3x^3}{4} - \frac{2x^3}{4} - \frac{x^3}{4} + \frac{2x^3}{4} + \frac{7x^3}{3}\right) + \left(\frac{6x^2}{4} + \frac{3x^2}{4} - \frac{14x^2}{3} - \frac{7x^2}{3} + \frac{13x^2}{2}\right)$$

$$+ \left(\frac{-6x}{4} + \frac{14x}{3} - \frac{13x}{2} + 12x\right)$$

$$= \frac{x^4}{4} + \frac{4x^2}{3} + \frac{7}{12}x^2 + \frac{26}{3}x$$

$\frac{x^3}{4} + \frac{6x^2+23x^2}{12}$, $\frac{3x^2+34x}{12}$, $\frac{16x^2}{12}$, $\frac{4x^2}{3}$

Date : Page :

Q. Evaluate $\Delta^2 \left\{ \frac{5x+12}{x^2+5x+6} \right\}$ June 2006

interval of differencing being unity

$\Delta^2 \left\{ \frac{5x+12}{x^2+5x+6} \right\}$

$= \Delta^2 \left\{ \frac{5x+12}{(x+2)(x+3)} \right\}$

$= \Delta^2 \left\{ \frac{2}{x+2} + \frac{3}{x+3} \right\}$

$= \Delta \left\{ \Delta \left(\frac{2}{x+2} + \frac{3}{x+3} \right) \right\}$

$= \Delta \left\{ 2\Delta \left(\frac{1}{x+2} \right) + 3\Delta \left(\frac{1}{x+3} \right) \right\}$

$\because \Delta f(x)$

$= f(x+h) - f(x)$

$= \Delta \left\{ 2 \left\{ \frac{1}{(x+2+1)} - \frac{1}{x+2} \right\} + 3 \left\{ \frac{1}{(x+3+1)} - \frac{1}{x+3} \right\} \right\}$

$= \Delta \left\{ 2 \left\{ \frac{1}{x+3} - \frac{1}{x+2} \right\} + 3 \left\{ \frac{1}{x+4} - \frac{1}{x+3} \right\} \right\}$

$= \Delta \left\{ 2 \left\{ \frac{x+2-x-3}{(x+2)(x+3)} \right\} + 3 \left\{ \frac{x+3-x-4}{(x+4)(x+3)} \right\} \right\}$

$= \Delta \left\{ 2 \cdot \frac{(-1)}{(x+2)(x+3)} + 3 \cdot \frac{-1}{(x+4)(x+3)} \right\}$

$= \Delta \left\{ \frac{-2}{(x+2)(x+3)} - \frac{3}{(x+4)(x+3)} \right\}$

Date : Page :

$$= -2\,\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}$$

$$= 2\left\{\frac{1}{(x+2+1)(x+3+1)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+3+1)(x+4+1)} - \frac{1}{(x+3)(x+4)}\right\}$$

$$= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}$$

$$= -2\left\{\frac{(x+2)(x+4)}{(x+2)(x+3)(x+4)}\right\} - 3\left\{\frac{(x+3)-(x+5)}{(x+3)(x+4)(x+5)}\right\}$$

$$= -2\left\{\frac{-2}{(x+2)(x+3)(x+4)}\right\} - 3\left\{\frac{(-2)}{(x+3)(x+4)(x+5)}\right\}$$

$$= \frac{4(x+5)+6(x+2)}{(x+2)(x+3)(x+4)(x+5}$$

$$= \frac{10x+32}{(x+2)(x+3)(x+4)(x+5)}$$

Q. Evaluate $\Delta^2\left\{\frac{1}{x(x+3)(x+6)}\right\}$ Interval of differencing being $h=3$.

$$\Delta\left\{\Delta\left(\frac{1}{x(x+3)(x+6)}\right)\right\}$$

$$= \Delta\left[\frac{1}{(x+3)(x+3+3)(x+6+3)} - \frac{1}{x(x+3)(x+6)}\right].$$

$$= \Delta\left\{\frac{1}{(x+3)(x+6)(x+9)} - \frac{1}{(x(x+3)(x+6)}\right\}$$

$$= \Delta \left[\frac{(x) - (x+9)}{x(x+3)(x+6)(x+9)} \right]$$

$$= -9\, \Delta \left[\frac{1}{x(x+3)(x+6)(x+9)} \right]$$

$$= -9 \left[\frac{1}{(x+3)(x+6)(x+9)(x+12)} - \frac{1}{x(x+3)(x+6)(x+9)} \right].$$

$$= -9 \left[\frac{-12}{x(x+3)(x+6)(x+9)(x+12)} \right].$$

$$= \frac{108}{x(x+3)(x+6)(x+9)(x+12)} \quad \text{Ans}$$

—x—

Interpolation - According to Theile.

"Interpolation is the art of reading between the lines of the table"

It is also means insertion or filling up intermediate terms of the series. Suppose we are to given the following values of $y = f(x)$ for a set of values of x:

x:	x_0	x_1	x_2	x_3	-	-	x_n
y:	y_0	y_1	y_2	y_3	-	-	y_n

Thus the process of finding the value of y corresponding to any value of $x = x_1$ between x_0 and x_n is called interpolation.
Hence interpolation is the technique of estimating the value of function for any intermediate value of the independent variable ~~xxxx~~ while the process of computing the value of the function outside the given range is called extrapolation.

~~Interp~~

(1) Interpoolation for Equal Intervals.

(A) Missing Value Method -

Q Estimate the missing term in the following table. Dec 2002, June - 2017 Dec 2009,

x :	0	1	2	3	4
$y=f(x)$:	1	3	9	-	81

Explain why value differs from 3^3 or 27.

Sol. First Method -

Here four entries are given so y can be represented by ~~third~~ Four degree polynomials.

$$\Delta^4 y = 0$$

$$(E-1)^4 y = 0 \quad \text{Expand binomial Expansion}$$

$$[E^4 - 4E^3 + 6E^2 - 4E + 1]y = 0$$

$$E^4 y - 4E^3 y + 6E^2 y - 4Ey + y = 0$$

$$f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x) = 0$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$81 - 4y_3 + 6(9) - 4(3) + 1 = 0$$

$$81 - 4y_3 + 54 - 12 + 1 = 0$$

$$4y_3 = 124$$

$$y_3 = \frac{124}{4} = 31$$

$\boxed{y_3 = 31}$ Hence at $x=3$, $y=31$

Since the interpolation based on polynomial function. But $3^3 = 27$ is not a polynomial because it is exponential function.

Second Method:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
		6		y_3-19	
2	9		y_3-15		$124-4y_3$
		y_3-9		$105-3y_3$	
3	y_3		$90-2y_3$		
		$81-y_3$			
4	81				

Here four values are known

$$\Delta^4 y = 0$$

$$124 - 4y_3 = 0$$

$$124 = 4y_3$$

$$y_3 = \frac{124}{4} = 31$$

$\boxed{y_3 = 31}$

Date : Page :

Q. Interpole $f(2.1)$ and $f(2.4)$ from the following function. June.2004

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	0.135	—	0.111	0.100	—	0.082	0.074

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.0	0.135					
		$y_1 - 0.135$				
2.1	y_1		$0.246 - 2y_1$			
		$0.111 - y_1$		$-0.368 + 3y_1$		
2.2	0.111		$-0.122 + y_1$		$y_4 + 0.401 - 4y_1$	
		-0.011		$y_4 + 0.033 - y_1$		$-0.183 - 5y_4 + 5y_1$
2.3	0.100		$y_4 - 0.089$.		$0.238 - 4y_4 + y_1$	
		$y_4 - 0.100$		$0.271 - 3y_4$.		$-0.781 + 10y_4 - y_1$
2.4	y_4		$0.182 - 2y_4$.		$-0.543 + 6y_4$.	
		$0.082 - y_4$.		$-0.272 + 3y_4$.		
2.5	0.082		$-0.09 + y_4$.			
		-0.008				
2.6.	0.074					

Since five values are given, ∴

Hence $\Delta^5 y = 0$

$$5y_1 - 5y_4 = 0.163.$$
$$-y_1 + 10y_4 = 0.781.$$

$$\boxed{\begin{aligned} y_1 &= 0.123 \\ y_4 &= 0.0904 \end{aligned}}$$

Date : Page :

Q. Find the missing term of the following table.

June 2003,
Dec 2007

x	1	2	3	4	5	6
y	–	8	3	0	-1	0.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	y_1					
		$8-y_1$				
2	8		y_1-13			
		-5		$15-y_1$		
3.	3		2		y_1-15	
		-3		0		$15-y_1$
4	0		2		0	
		-1		0		
5	-1		2			
		1				
6	0					

Here five values are given

Since $\Delta^5 y = 0$

$$15 - y_1 = 0$$

$$y_1 = \boxed{15}$$

A

Q Compute the Next three Values: Dec 2004

x	0	1	2	3	4	5	6	7
$y=f(x)$	1	-1	1	-1	1	–	–	–

Sol

Dec 2004

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		-2				
1	-1		4			
		2		-8		
2	1		-4		16	
		-2		8		y_5-31
3	-1		4		y_5-15	
		2		y_5-7		y_6-5y_5+26
4	1		y_5-3		y_6-4y_5+11	
		y_5-1		y_6-3y_5+4		$y_7-5y_6+10y_5-16$
5	y_5		y_6-2y_5+1		$y_7-4y_6+6y_5-5$	
		y_6-y_5		$y_7-3y_6+3y_5-1$		
6	y_6		$y_7-2y_6+y_5$			
		y_7-y_6				
7	y_7					

Hence five values are given $\Delta^5 y = 0$

Since
$$0 + 0 + y_5 = 31$$
$$0 + y_6 - 5y_5 = -26$$
$$y_7 - 5y_6 + 10y_5 = 16$$

$$\left.\begin{aligned} y_7 &= 351 \\ y_6 &= 129 \\ y_5 &= 31 \end{aligned}\right\} \text{Ans}$$

Date : Page :

* Interpolation for Equal Interval

(B) Intermediate Value Method.

Method-1:
Newton's Gregory Forward Difference Interpolation Formula.

Suppose $y = f(x)$ by any function

x	x_0	x_0+h	x_0+2h	-----	x_0+nh
$y=f(x)$	$f(x_0)$	$f(x_0+h)$	$f(x_0+2h)$	-----	$f(x_0+nh)$

and $p = \dfrac{x-x_0}{h}$

where, x = estimate value
x_0 = Initial value
h = interval

then, $y_x = f(x_0+ph) = y_0 + pc_1\,\Delta y_0 + pc_2\,\Delta^2 y_0 + pc_3\,\Delta^3 y_0 + \cdots$

or

$$f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots$$

Method -2:
Newton's Gregory Backward Difference Interpolation Formula.

Date : Page :

Suppose $y = f(x)$ be any function

x	x_0	x_0+h	x_0+2h	x_0+3h	- - - -	x_0+nh
$y=f(x)$	$f(x_0)$	$f(x_0+h)$	$f(x_0+2h)$	$f(x_0+3h)$	- - -	$f(x_0+nh)$

and $p = \dfrac{x - x_n}{h}$

where, x = estimate value

x_n = last value of x.

h = interval .

$$f(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \cdots$$

NOTE :-

(i) If estimate value near the begining of the table, then we use Newton Forward method.

(ii) If the estimate value near the end of the table, then we use Newton Backward method.

Rough

Forward Difference

$$f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

$$f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots$$

Backward

$$f(x_n+ph) = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \cdots$$

Q. Given,

$y_{20} = 24$, $y_{24} = 32$

$y_{28} = 35$, $y_{32} = 40$.

Find y_{22} by Newton Gregory Forward Difference interpolation formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$.
20	24			
		8		
			-5	
24	32			7
		3		
			2	
28	35			
		5		
32	40			

Newton forward formula

$$f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

Here, $p = \frac{x-x_0}{h} = \frac{22-20}{4} = 0.5$

$$f(x_0+ph) = 24 + (0.5\times 8) + \frac{0.5(0.5-1)}{2}\times -5 + \frac{0.5(0.5-1)(0.5-2)}{3\times 2}\times 7$$

$f(20+0.5\times 4) = 29.0625$

$f(22) = 29.0625$.

Q Given

x :	0	1	2	3
y :	1	2	1	10

Find the cubic polynomial which take the following value. Hence or otherwise evaluate $y(4)$.

Dec 2001, Feb. 2010, Dec 2014, June-2017, Nov-2018.

$$
\begin{array}{ccccc}
x & y & \Delta y & \Delta^2 y & \Delta^3 y \\
0 & 1 & & & \\
 & & 1 & & \\
1 & 2 & & -2 & \\
 & & -1 & & 12 \\
2 & 1 & & 10 & \\
 & & 9 & & \\
3 & 10 & & &
\end{array}
$$

$$f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

Here, $p = \frac{x-x_0}{h} = x.$

$$f(0+x(1)) = f(x) = 1 + x(1) + \frac{x(x-1)}{\lfloor 2}(-2) + \frac{x(x-1)(x-2)\times(12)}{\lfloor 3}$$

$$f(x) = 2x^3 - 7x^3 + 6x + 1.$$

$$f(4) = 41.$$

Q. The pressure p of wind corresponding to velocity v is given by following data. Estimate p when $v=15$

June - 2002

v	10	20	30	40
p	1.1	2.0	4.4	7.9.

V	p	Δp	$\Delta^2 p$	$\Delta^3 p$
10	1.1			
		0.9		
20	2.0		1.5	
		2.2		-0.4
30	4.4		1.1	
		3.5		
40	7.9			

$$p = \frac{x - x_0}{h} = \frac{15-10}{10} = 0.5$$

$$f(10 + 0.5 \times 10) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

$$f(15) = 1.1 + (0.5 \times 0.9) + \frac{0.5(0.5-1)}{2}(1.5) + \frac{0.5(0.5-1)(0.5-2)}{3 \times 2}$$

$$f(15) = \underline{\underline{1.3375}}$$

Q. Estimate from the table the no. of students who obtained marks between 40 and 45.

marks	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
no. of students.	31	42	31	35	31

Dec 2001, Dec 2012, Dec 2017

Date : Page :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		+12	
70	159		-4		
		31			
80	190				

$$p = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5$$

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\underline{|2}} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\underline{|3}} \Delta^3 y_0 +$$

$$\frac{p(p-1)(p-2)(p-3)}{24} \Delta^4 y_0$$

$$f(40 + 0.5 \times 10) = 31 + (0.5 \times 42) + \frac{0.5(0.5-1) \times 9}{2} + \frac{0.5(0.5-1)(0.5-2)(-25)}{3 \times 2}$$

$$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3) \times (37)}{4 \times 3 \times 2}$$

$$f(45) = 31 + 21 + (-1.125) + (-1.5625) + (-1.4453)$$
$$f(45) = 47.8671875 \approx 48$$

No. of students whose marks is in between 40-45

i.e., $f(45) - f(40) = 48 - 31 = 17$.

Date : Page :

Q Find the no. of man getting wages between 10 and 15 rs. from the following table.

Dec. 2013

wages (x)	0-10	10-20	20-30	30-40
frequency (x)	9	30	35	42

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$,
10	9			
		30		
20	39		5	
		35		2.
30	74		7	
		42		
40.	116.			

$$p = \frac{x - x_0}{h} = \frac{15-10}{10} = \frac{5}{10} = 0.5$$

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\underline{|2}} \Delta^2 y + \frac{p(p-1)(p-2)}{\underline{|3}} \Delta^3 y.$$

$$f(10+0.5(10)) = 9 + (0.5 \times 30) + \frac{(0.5(0.5-1)(5)}{\underline{|2}} + \frac{0.5(0.5-1)(0.5-2)\times(2)}{\underline{|3}}$$

$$f(15) = 9 + 15 + (-0.625) + (0.125)$$

$$= 23.5$$

$$\boxed{f(15) = 24.}$$

No. of man getting wages betw 10 and 15

$= f(15) - f(10)$

$= 24 - 9 = \underline{15.}$

Date : Page :

Q. Evaluate $f(42)$ from the table.

x	20	25	30	35	40	45
y	354	332	291	260	231	204

June - 2006

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
20	354					
		-22				
			-19			
25	332			29		
		-41			-37	
30	291		10			45
		-31		-8		
					8	
35	260		2			
		-29		0		
40	231		2			
		-27				
45	204					

$$p = \frac{x - x_n}{h} = \frac{42-45}{5} = \frac{-3}{5} = -0.6$$

$$f(x_n+ph) = y_n + p\nabla y_n + \frac{p(p+1)}{\underline{|2}} \nabla^2 y_n + \frac{p(p+1)(p+2)}{\underline{|3}} \nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{\underline{|4}} \nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{\underline{|5}} \nabla^5 y_n$$

$$f(42) = 204 + (-0.6 \times -27) + \frac{(-0.6)(-0.6+1)}{2} \times 2 +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)}{3 \times 2} \times 0 +$$

Date : Page :

$$\frac{-(0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4\times3\times2}\times 8 \;+$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5\times4\times3\times2}\times 45$$

$= 204 + 16.2 + (-0.24) + 0 + (-0.2688) + (-1.02816)$

$\boxed{f(42) = 218.66304}$

Q.

x	1931	1941	1951	1961	1971	1981
y	12	15	20	27	39	52

June - 2003

Evaluate f (1966) from the table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1931	12					
		3				
1941	15		2			
		5		0		
1951	20		2		3	
		7		3		-10
1961	27		5		-7	
		12		-4		
1971	39		1			
		13				
1981	52					

Date : Page :

$x = 1966$

$x_n = 1981$

$h = 10$

$$p = \frac{x - x_n}{h} = \frac{1961 - 1981}{10} = \frac{-15}{10} = -1.5$$

$$f(x_n + ph) = y_n + p\nabla y + \frac{p(p+1)}{\lfloor 2} \nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3} \nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{\lfloor 4} \nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{\lfloor 5} \nabla^5 y_n$$

$$f(1966) = 52 + (-1.5 \times 13) + \frac{(-1.5)(-1.5+1)}{2} \times 1$$

$$+ \frac{(-1.5)(-1.5+1)(-1.5+2)}{3 \times 2} \times (-4)$$

$$+ \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)}{4 \times 3 \times 2} \times (-7)$$

$$+ \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5 \times 4 \times 3 \times 2} \times (-10)$$

$$f(1966) = 52 + (-19.5) + (0.375) + (-0.25) + (-0.1640625)$$
$$+ (-0.1171875)$$

$$\boxed{f(1966) = 32.34375}$$

Date : Page :

* Interpolation for Unequal Intervals

Method-(I)

1. Lagrange's Interpolation formula.

If $y = f(x)$, takes the values $y_0, y_1, y_2, y_3 --- y_n$ corresponding to $x = x_0, x_1, x_2 --- x_n$.

then,

$$f(x) = \frac{(x-x_1)(x-x_2)----(x-x_n)}{(x_0-x_1)(x_0-x_2)---(x_0-x_n)}(y_0) +$$

$$\frac{(x-x_0)(x-x_2)---(x-x_n)}{(x_1-x_0)(x_1-x_2)---(x_1-x_n)}(y_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)---(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)---(x_2-x_n)}(y_2) + ---$$

Q. Use Lagrange's Formula to find the value of y when $x = 10$. If the values of x and y are given as below.

x	5^{x_0}	6^{x_1}	9^{x_2}	11^{x_3}
y	12^{y_0}	13^{y_1}	14^{y_2}	16^{y_3}

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$f(10) = \frac{4\times1\times(-1)}{(-1)(-4)(-6)}(12) + \frac{5\times1\times(-1)}{1\times(-3)(-5)}(13) + \frac{5\times4\times(-1)}{4\times3\times(-2)}\times14$$

$$+ \frac{(5)(4)(1)(16)}{6\times5\times2} = 2 + \frac{-13}{3} + \frac{35}{3} + \frac{16}{3} = 14.66$$

Date : Page :

Q. Given the value June-2011, Dec 2013, June-2014

x	5	7	11	13	17
y	150	392	1452	2368	5202

Evaluate f(9) by using ~~QJe formula~~ Lagrange's Formula.

$$f(10) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150$$

$$+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2368$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$f(9) = \frac{2\times(-2)\times(-4)\times(-8)}{(-2)(-6)(-8)(-12)} \times 150 + \frac{4\times(-2)(-4)(-8)}{2\times(-4)\times(-6)\times(-10)} \times 392$$

$$+ \frac{4\times2\times(-4)(-8)}{6\times4\times(-2)(-6)} \times 1452 + \frac{4\times2\times(-2)(-8)}{8\times6\times2\times(-4)} \times 2368$$

$$+ \frac{4\times2\times(-2)(-4)}{12\times10\times6\times4} \times 5202$$

$$= \frac{-50}{3} + \frac{3136}{15} + \frac{3872}{3} + \frac{-2368}{3} + \frac{578}{5}$$

$$\boxed{f(9) = 809.33}$$

Q. Use Lagranges formula to find the form of $f(x)$ given. June 2006, Dec 2008.

x	0	2	3	6
$f(x)$	648	704	729	792

$$f(x) = \frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)} \times 648 + \frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)} \times 704$$

$$+ \frac{(x-0)(x-2)(x-6)}{(3-0)(3-2)(3-6)} \times 729 + \frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)} \times 792$$

$$= \frac{(x-2)(x-3)(x-6)}{\cancel{-36}} \times \overset{-18}{\cancel{648}} + \frac{x(x-3)(x-6)}{\cancel{8}} \times \overset{88}{\cancel{704}}$$

$$+ \frac{x(x-2)(x-6)}{\cancel{-9}} \times \overset{-81}{\cancel{729}} + \frac{x(x-2)(x-3)}{\cancel{72}} \times \overset{11}{\cancel{792}}$$

$$= (x-2)(x-3)(x-6)(-18) + x(x-3)(x-6)(-88)$$
$$+ x(x-2)(x-6)(-81) + x(x-2)(x-3)(11)$$

$$\cancel{= f(x^2-3x)(-2x+6)}$$
$$= (x^2-3x-2x+6)(x-6)(-18) + (x^2-3x)(x-6)(-88)$$
$$+ (x^2-2x)(x-6)(-81) + (x^2-2x)(x-3)(11)$$

$$\cancel{= (x^3-6x^2+18x+12x-36)(-18) + x^3-6x^2+}$$
$$= x^3-3x^2-2x^2+6x-6x^2+18x+12x-36)(-18)$$
$$+ (x^3-6x^2-3x^2+18x)(-88) + (x^3-6x^2-2x^2+12x)(-81)$$
$$+ (x^3-3x^2-2x^2+6x)(11)$$
$$= -18x^2+54x^2+36x^2-108x+108x^2-324x-216x+648$$
$$+ 88x^3+528x^2+264x^2-1584x+81x^3+486x^2+162x$$

Date : Page :

$- \cancel{972x + 11x^3 - 33x^2 - 22x^2 + 66x}$

=

$f(x) = -x^2 + 30x + 648$

Method (II)

Q. ~~Lagrange~~ Newton's Divided Difference formula.

$$f(x) = f(x_0) + (x - x_0)\, \Delta f(x_0) + (x - x_0)(x - x_1)\, \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\, \Delta^3 f(x_0) + \cdots$$

OR.

$$y = y_0 + (x - x_0)\, \Delta y + (x - x_0)(x - x_1)\, \Delta^2 y + (x - x_0)(x - x_1)(x - x_2)\, \Delta^3 y + \cdots$$

Q. By using Newton Divided difference formula, find the value of $f(8)$, $f(9)$, $f(15)$ for the table –

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

* Dec 2006, June-2012, Dec 2012, Nov-2018

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
4	$\boxed{48}$					
		$\frac{100-48}{5-4} = \boxed{52}$				
			$\boxed{15}$			
5	100					
		$\cancel{194}$ 97		$\boxed{1}$		
7	294		21		$\frac{1-1}{11-4} = \frac{0}{8} = \boxed{0}$	
		$\cancel{606}$ 202		1		$\frac{0-0}{13-4} = \boxed{0}$
			27		$\frac{1-1}{13-5} = \frac{0}{8} = 0$	
10	900					
		310		1		
11	1210		33			
		$\cancel{818}$ 409				
13	2028					

~~$y = y_0$~~

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + \ldots$$

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + \ldots$$

$$= 48 + (x-4)52 + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1)$$

~~$= 48$~~ $+$

$$f(8) = 48 + (8-4)52 + (8-4)(8-5)(15) + (8-4)(8-5)(8-7)(1)$$

$$= 48 + 208 + 180 + 12$$

$$= 448.$$

$$f(9) = 48 + (9-4)52 + (9-4)(9-5)(15) + (9-4)(9-5)(9-7)(1)$$

$$= 48 + 260 + 300 + 40$$

$$= 648.$$

$$f(15) = 48 + (15-4)52 + (15-4)(15-5)(15) + (15-4)(15-5)(15-7)(1)$$

$$= 48 + 572 + \overset{2400}{\cancel{1650}} + 880$$

$$= 3150.$$

$34 - 11 = 23$

Date : Page :

Q. Use Newton divided diff. formula find $f'(10)$

x	3	5	11	27	34
$f(x)$	-13	23	899	17315	35606

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	-13				
		18			
5	23		16		
		146		1	
11	899		40		0
		1026		1	
27	17315		69		
		2613			
34	35606				

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \cdots$$

$$= y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + \cdots$$

$$= -13 + (x-3)(18) + (x-3)(x-5)(16) + (x-3)(x-5)(x-11)(1)$$

$$= -13 + 18x - 54 + (x^2 - 5x - 3x + 15)16 + (x^2 - 5x - 3x + 15)(x-11)(1)$$

$$f(x) = x^3 - 3x^2 - 7x + 8.$$

$$f'(x) = 3x^2 - 6x - 7.$$

$$f'(10) = 3(10)^2 - 60 - 7$$

$$= 300 - 67$$

$$= 233 \text{ Ans}$$

Q. Find a polynomial satisfied by $(-4, 1245)$, $(-1, 33)$, $(0, 5)$, $(2, 9)$ and $(5, 1335)$ by Newton divided difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + \text{---}$$

$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x)(-14)$$
$$+ (x+4)(x+1)(x)(x-2)(3)$$

$$= 1245 - 404x - 1616 + (x^2 + x + 4x + 4)(94) + (x^3 + x^2 + 4x^2 + 4x)(-14)$$
$$+ (x^3 + x^2 + 4x^2 + 4x)(x-2)(3)$$
$$= 1245 - 404x - 1616 + 94x^2 + 94x + 376x + 376 - 14x^3 - 14x^2 - 56x^2 - 56x$$
$$+ (x^4 + x^3 + 4x^3 + 4x^2 - 2x^3 - 2x^2 - 8x^2 - 8x)3$$

$$= 1245 - 404x - 1616 + 94x^2 + 94x + 376x + 376 - 14x^3 - 14x^2$$
$$- 56x^2 - 56x + 3x^4 + 3x^3 + 12x^3 + 12x^2 - 6x^3 - 6x^2 - 24x^2 - 24x$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$