# LAKSHMI NARAIN COLLEGE OF TECHNOLOGY, BHOPAL

## DEPARTMENT OF ELECTRICAL & ELECTRONICS ENGINEERING

**SUBJECT: CONTROL SYSTEM (EX-405)**

**New Scheme Based On AICTE Flexible Curricula of Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal**

# Study Material for Unit III- Stability Analysis

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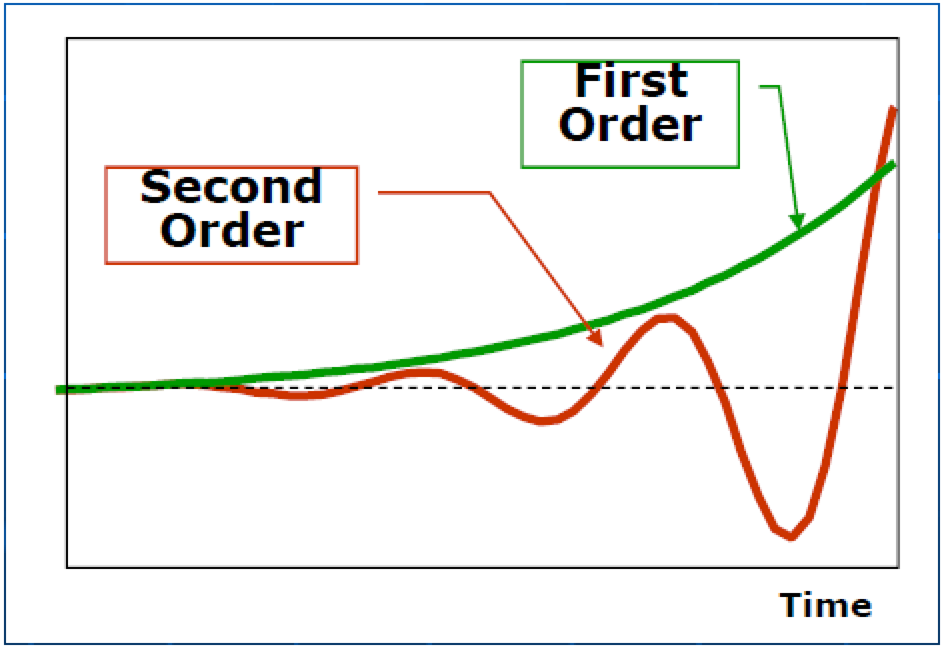
# Definition of Stability

- The ability of any given system to attain the steady state condition after passing through transients successfully is called stability.
- **Bounded-Input, Bounded-Output (BIBO) Stability:**

➢ A linear time invariant system is said to be stable if it produces a bounded response to a bounded input.

➢ Thus for an unstable system, the response will increase without bounds or will never return to the equilibrium state.




From the fig. it can be clearly seen that for both first and second order systems the response w.r.t. time is increasing rather than settling down, thus showing an unstable condition .

# TRANSFER FUNCTION

- General form of the transfer function :

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

n : order of the system ($n \geq m$ ),

D(s) : characteristic polynomial

Characteristic equation : D(s)=0

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

- The roots of the numerator polynomial, i.e.

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0 = 0$$

are called the zeroes of the system.

- The roots of the denominator polynomial are called the poles of the system.

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

# STABILITY AND POLES

Let us take an example: Consider a closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{10}{(s+2)(s+4)}$$

Let R(s)=1/s, thus obtaining the response for unit step input

$$C(s) = \frac{1.25}{s} - \frac{2.5}{s+2} + \frac{1.25}{s+4}$$

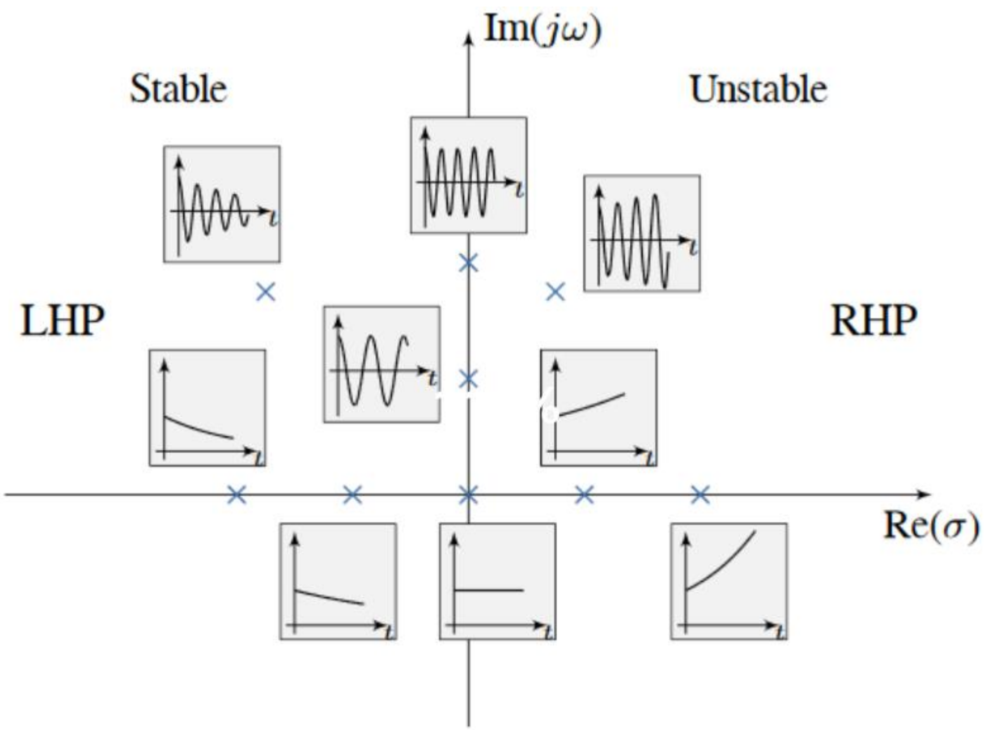
In time domain the response is given by:

$$c(t) = 1.25 - 2.5e^{-2t} + 1.25e^{-4t}$$

when t→∞, it can be seen that both the exponential terms will approach to zero and the output will be steady state output of 1.25.

- ❖ **On observing the given transfer function it can be seen that the poles of the transfer function are located at s=-2 and s=-4 i.e. on the left hand side of the s-plane.**
- ❖ **If the poles are located in the LHS of the s-plane, exponential indices in output are negative, hence the exponential transient terms will vanish when t→∞. Thus, making the system absolutely stable.**
- ❖ **On the contrary, if there are poles in the RHS of s-plane then the exponential terms will tend to unbounded values which will not return to its steady state values and the system becomes absolutely unstable.**

- **Stability or instability is a property of the system itself i.e. closed loop poles of the system and does not depend on input or driving function. The poles of input do not affect stability of the system they affect only steady state output**
- **If all the roots of the characteristic equation are on the left hand side of the complex plane, i.e. all the roots have negative real parts, then the system is stable.**
- **If there is at least one root on the right hand side of the complex plane, then the system is unstable and the response will increase without bounds with time.**
- **If there is at least one root with zero real part, i.e. on the imaginary axis, then the response will contain undamped sinusoidal oscillations or a non decaying response.**



# ROUTH'S STABILITY CRITERION

- Routh's stability criterion allows the determination of
- whether there are any roots of the characteristic equation with positive real parts

and, if there are,

- the number of these roots

without actually finding the roots.

- The first step in checking the stability of a system using Routh's stability criterion is the application of an initial test called the Hurwitz test.

<u>Hurwitz Test :</u>

- The necessary but not sufficient condition for a characteristic equation

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

to have all its roots with negative real parts is that all of the coefficients $a_i$ must exist and have the same sign.

- If the characteristic equation fails to meet the above condition, then the system is not stable.

Eg: $D(s) = 3s^4 + 2s^3 + s^2 + 5$

Here the power of s and its coefficient is missing, hence unstable.

- A sufficient condition for instability All signs of the coefficients of the characteristic polynomial are not the same → unstable system

# ROUTH'S STABILITY CRITERION

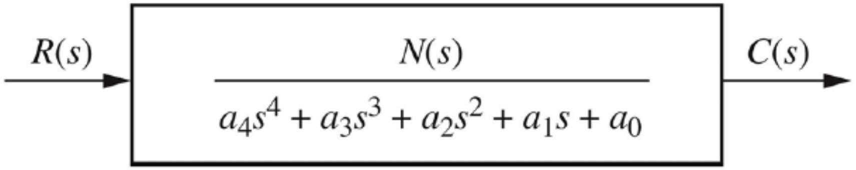
- If Hurwitz condition is satisfied, then Routh's stability criterion must be used to determine the stability of the system.
- To be able to apply Routh's criterion, Routh's array must be constructed.
- For a real polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

the Routh's array is a special arrangement of the coefficients in a certain pattern.

# CONSTRUCTING A BASIC ROUTH TABLE

- **Begin by labelling the rows with powers of s from the highest power of the denominator polynomial to $s_0$**
- **List in the first row every other coefficient starting with the one of the highest power of s**
- **List in the second row coefficients that were skipped in the first row**
- **Complete the rest of the table**




Initial layout for Routh table

| | | | |
|---|---|---|---|
| $s^4$ | $a_4$ | $a_2$ | $a_0$ |
| $s^3$ | $a_3$ | $a_1$ | 0 |
| $s^2$ | | | |
| $s^1$ | | | |
| $s^0$ | | | |

# CONSTRUCTING A BASIC ROUTH TABLE

## Routh array
## (How to compute the third row)

| | | | | | |
|---|---|---|---|---|---|
| $s^n$ | $a_n$ | $a_{n-2}$ | $a_{n-4}$ | $a_{n-6}$ | $\cdots$ |
| $s^{n-1}$ | $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $a_{n-7}$ | $\cdots$ |
| $s^{n-2}$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $\cdots$ |
| $s^{n-3}$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | | |
| $s^2$ | $k_1$ | $k_2$ | | | |
| $s^1$ | $l_1$ | | | | |
| $s^0$ | $m_1$ | | | | |

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$\vdots$$

# CONSTRUCTING A BASIC ROUTH TABLE

## Routh array
## (How to compute the fourth row)

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

$$
\begin{aligned}
c_1 &= \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} \\
c_2 &= \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} \\
&\vdots
\end{aligned}
$$

# ROUTH HURWITZ CRITERION

## Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

*The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.*

# ROUTH HURWITZ CRITERION

- **Example :** $D(s)=s^3+20s^2+9s+100$

**Passes Hurwitz test !**

| | | |
|---|---|---|
| $s^3$ | 1 | 9 |
| $s^2$ | 20 | 100 |
| $s^1$ | $\boxed{4}$ | |
| $s^0$ | $\boxed{100}$ | |

$$b_1 = \frac{(20)(9)-(1)(100)}{20} = 4$$

$$c_1 = \frac{(4)(100)-(20)(0)}{4} = 100$$

**Knight's move (chess)**

# ROUTH HURWITZ CRITERION

- **The necessary and sufficient condition for a characteristic equation to have all its roots with negative real parts is that the elements of the first column of the Routh's array to have the same sign.**
- **If the elements of the first column have different signs, then the number of sign changes is equal to the number of roots with positive real parts.**

- **Example :** $D(s)=s^3+s^2+2s+24$

Passes Hurwitz test !

$s_1 = -3.0000$
$s_2 = 1.0000 + 2.6458i$
$s_3 = 1.0000 - 2.6458i$

| | | |
|---|---|---|
| $s^3$ | 1 | 2 |
| $s^2$ | 1 | 24 |
| $s^1$ | -22 | 0 |
| $s^0$ | 24 | |

$$b_1 = \frac{(1)(2)-(1)(24)}{1} = -22$$

$$c_1 = \frac{(-22)(24)-(1)(0)}{-22} = 24$$

Knight's move

Sign changes in the 1st column : Unstable system.

2 sign changes : two roots with positive real parts.

# ROUTH HURWITZ CRITERION

- **Special Cases :**

There are some cases in which problems appear in completing the Routh's array.

- **Special Case 1 :**

When a first column term in a row becomes zero with all other terms being nonzero, the calculation of the rest of the terms becomes impossible due to division by zero. In such a case the system is unstable and the procedure is continued just to determine the number of roots with positive real parts.

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 2 | 11 |
| $s^4$ | 2 | 4 | 10 |
| $s^3$ | 0 | 6 | 0 |
| $s^2$ | ??? | | |
| $s^1$ | | | |
| $s^0$ | | | |

$$b_1 = \frac{(2)(2)-(1)(4)}{2} = 0$$

$$b_2 = \frac{(2)(11)-(1)(10)}{2} = 6$$

$$c_1 = \frac{(0)(4)-(2)(6)}{0}$$

# ROUTH HURWITZ CRITERION

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 2 | 11 |
| $s^4$ | 2 | 4 | 10 |
| $s^3$ | 0 | 6 | 0 |
| $s^2$ | ??? | | |
| $s^1$ | | | |
| $s^0$ | | | |

In such a case, replace zero term by a very small and positive number $\varepsilon$.

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 2 | 11 |
| $s^4$ | 2 | 4 | 10 |
| $s^3$ | $0 \rightarrow \varepsilon$ | 6 | 0 |
| $s^2$ | $-\frac{12}{\varepsilon}$ | 10 | 0 |
| $s^1$ | 6 | 0 | |
| $s^0$ | 10 | | |

$$c_1 = \frac{(4)(\varepsilon)-(2)(6)}{\varepsilon} = 4-\frac{12}{\varepsilon}$$

$$\varepsilon \rightarrow 0 \quad \Rightarrow \quad c_1 \cong -\frac{12}{\varepsilon}$$

$$d_1 = \frac{\left(-\frac{12}{\varepsilon}\right)(6)-(10)(\varepsilon)}{-\frac{12}{\varepsilon}} = 6+\frac{10}{12}\varepsilon^2$$

$$\varepsilon \rightarrow 0 \quad \Rightarrow \quad d_1 \cong 6$$

# ROUTH HURWITZ CRITERION

- **Special Case 1 :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 2 | 11 |
| $s^4$ | 2 | 4 | 10 |
| $s^3$ | $0 \rightarrow \varepsilon$ | 6 | 0 |
| $s^2$ | $-\dfrac{12}{\varepsilon}$ | 10 | |
| $s^1$ | 6 | | |
| $s^0$ | 10 | | |

**2 sign changes :**

2 roots with positive real part.

$$s_1 = 0.8950 + 1.4561i$$
$$s_2 = 0.8950 - 1.4561i$$
$$s_3 = -1.2407 + 1.0375i$$
$$s_4 = -1.2407 - 1.0375i$$
$$s_5 = -1.3087$$

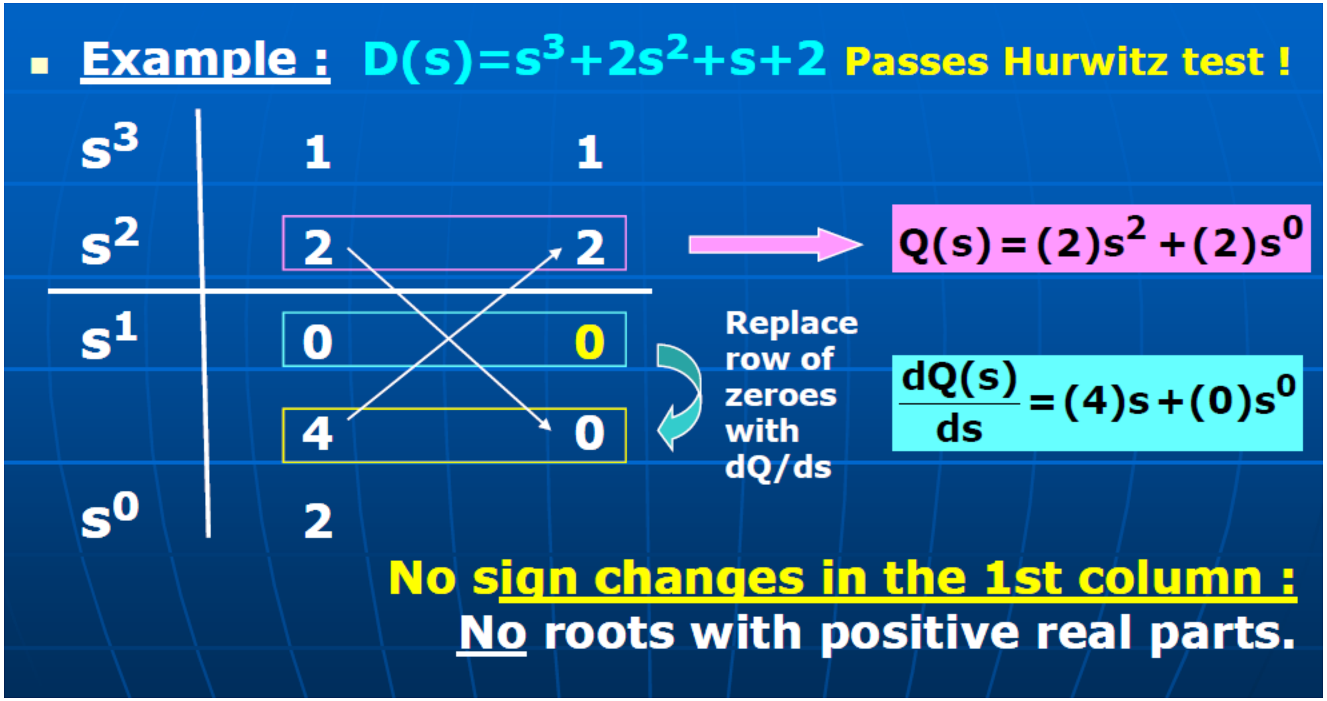
# ROUTH HURWITZ CRITERION

- **Special Case 2 :**

If all the terms on a derived row are zero, this means that the characteristic equation has roots which are symmetric with respect to the origin,

1. Two real roots with equal magnitudes but opposite signs, and/or
2. Two conjugate imaginary roots, and/or
3. Two complex roots with equal real and imaginary parts of opposite signs.

- To proceed, an auxiliary polynomial Q(s) is formed by using the terms of the row just before the row of zeros. The auxiliary polynomial Q(s) is always even (i.e. all powers of s are even).
- The roots of Q(s)=0 will give the symmetric roots of the characteristic polynomial.
- To complete the Routh's array, simply replace the row of zeroes with the coefficients of dQ(s)/ds=0.



# ROUTH HURWITZ CRITERION

- In order to find the complete stability of the previous system, no. of imaginary roots should also be identified.
- For this, auxiliary equation is equated to zero and roots are obtained, as shown below. Thus, location of all the roots can be identified.

- **Example :** $D(s)=s^3+2s^2+s+2$ **Passes Hurwitz test !**

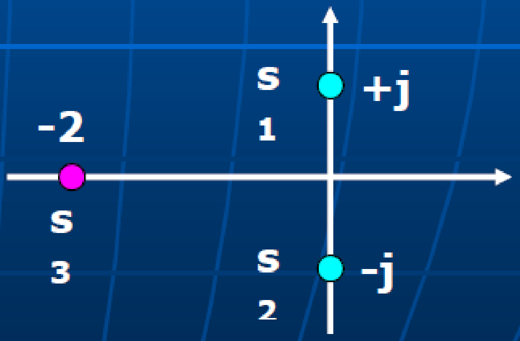
| | | |
|---|---|---|
| $s^3$ | 1 | 1 |
| $s^2$ | 2 | 2 |
| $s^1$ | 0 | 0 |
| | 4 | 0 |
| $s^0$ | 2 | |

$$Q(s) = 2s^2 + 2 = 0$$

$$s_{1,2} = \pm j$$

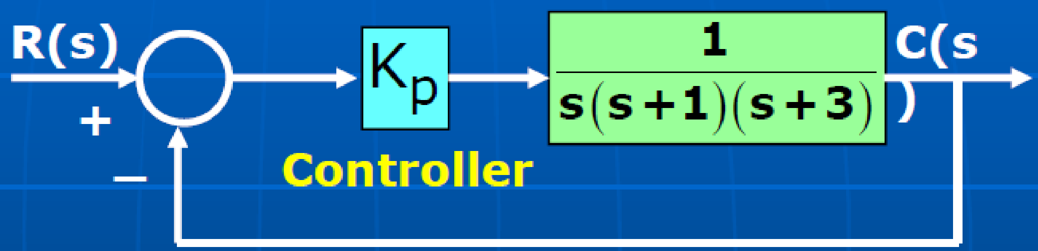
$$\frac{s^3+2s^2+s+2}{2s^2+2} = \frac{s}{2}+1$$

$$s_3 = -2$$



# STABILITY BOUNDARIES

- One of the steps in the design and optimization of control systems is the selection of controller parameters.
- The limiting values of these parameters leading to instability must be determined first, so that best values in the stable range can be chosen.
- Eg:
  - Determine the range of values for the controller parameter $K_p$ for which the system will be stable.




  - First determine the characteristic polynomial.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K_p}{s(s+1)(s+3)}}{1+\dfrac{K_p}{s(s+1)(s+3)}} = \frac{K_p}{s(s+1)(s+3)+K_p} = \frac{K_p}{s^3+4s^2+3s+K_p}$$

$$\Longrightarrow \quad D(s)=s^3+4s^2+3s+K_p$$

# STABILITY BOUNDARIES

- $D(s) = s^3 + 4s^2 + 3s + K_p$

| | | |
|---|---|---|
| $s^3$ | 1 | 3 |
| $s^2$ | 4 | $K_p$ |
| $s^1$ | $3 - \frac{K_p}{4}$ | 0 |
| $s^0$ | $K_p$ | |

**For stability :**

$$3 - \frac{K_p}{4} > 0 \quad \text{and} \quad K_p > 0$$

$$0 < K_p < 12$$

# STABILITY BOUNDARIES

- $D(s)=s^3+4s^2+3s+K_p$

| | | |
|---|---|---|
| $s^3$ | 1 | 3 |
| $s^2$ | 4 | $K_p$ |
| $s^1$ | $3-\frac{K_p}{4}$ | 0 |
| $s^0$ | $K_p$ | |

For $K_p = 0$:

$s(s+1)(s+3)=0$

$s_1=0,\ s_2=-1,\ s_3=-3$

Marginally stable!

Non-oscillatory response.

For $K_p = 12$:

$(s+4)(s^2+3)=0$

$s_1=-4,\ s_2=+\sqrt{3}j,\ s_3=-\sqrt{3}j$

Marginally stable!

Oscillatory response,

Undamped oscillations.

**STABILITY BOUNDARIES**

## Marginal K and Frequency of Sustained Oscillations

- Marginal value of ‘k’ is that value for which system becomes marginally stable.
- For marginal stability there must be a row of zeros occurring in Routh’s Array. So, value of k which makes any row of Routh’s Array as row of zeros is called marginal value of ‘k’.
- To obtain the frequency of oscillations, solve the auxiliary equation A(s)=0 for $K=K_{mar}$ the magnitude of imaginary roots of A(s)=0 obtained for marginal value of k indicates the frequency of sustained oscillations which system will produce.

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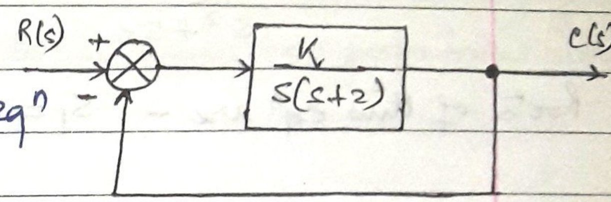
# ROOT LOCUS TECHNIQUE

- The locus of the roots of the characteristic eq$^n$ when gain is varied from zero to infinity is called Root Locus.

- Consider a unity feedback system as shown in fig. The characteristic eq$^n$ is $1 + G(s)\,H(s) = 0$



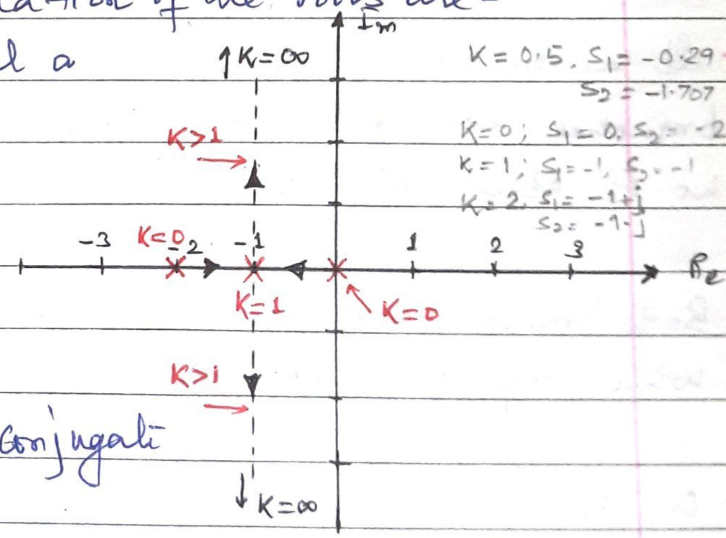

where $G(s) = \dfrac{K}{s(s+2)}$ and $H(s) = 1$

$\therefore \quad 1 + \dfrac{K}{s(s+2)} \cdot 1 = 0 \quad$ or $\quad s^2 + 2s + K = 0$ — (1)

Roots of eq$^n$(1) are – $s_1 = -1 + \sqrt{1-K}$, $s_2 = -1 - \sqrt{1-K}$

- On varying the value of 'K', the two roots give the locii in s-plane. For various values of 'K', the location of the roots are–

1) when $0 < K < 1$, the roots are real and distinct
2) when $K = 0$, the roots are $s_1 = 0$ and $s_2 = -2$
3) when $K = 1$, both roots are real and equal.
4) when $K > 1$, roots are complex conjugate with real part $= -1$




- When K is varying the root locus is shown in above figure.
1) when $K = 0$, two branches of the root locus starts from $s=0$ & $s=-2$
2) when $K = 1$, both the roots meet at $s = -1$
3) when $K > 1$, the roots breakaway from the real axis and become complex conjugate having -ve real part equal to $-1$.

- No. of branches is equal to the no. of open loop poles.

Consider $G(s)\,H(s) = \dfrac{K(s+1)}{s(s+5)}$ and obtain the nature of root locus.

- Characteristic eq$^n$ is – $1 + G(s)H(s) = 0$

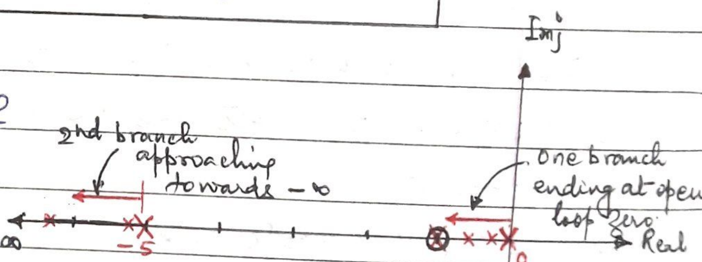
  i.e. $1 + \dfrac{K(s+1)}{s^2+5s} = 0$ or $s^2 + (5+K)s + K = 0$

Roots of this eq$^n$ are – $s_1 = -\dfrac{(K+5)}{2} + \dfrac{\sqrt{K^2+6K+25}}{2}$ & $s_2 = -\dfrac{(K+5)}{2} - \dfrac{\sqrt{K^2+6K+25}}{2}$

- Effect of variation of K –

| K | $s_1 = -\dfrac{(K+5)}{2} + \dfrac{\sqrt{K^2+6K+25}}{2}$ | $s_2 = -\dfrac{(K+5)}{2} - \dfrac{\sqrt{K^2+6K+25}}{2}$ |
|---|---|---|
| 0 | 0 | −5 |
| 1 | −0.1715 | −5.828 |
| 5 | −0.527 | −9.472 |
| ⋮ | | |
| ∞ | −1 | −∞ |

- It can be observed that no. of branches are again two i.e. the no. of open loop poles.




- Both branches are starting from $s=0$ and $s=-5$ which are open loop poles.

- One of the important observations is that, one of the branches terminates at $s=-1$, which is an open loop zero, while the other branch is terminating at infinity.

- It is difficult to plot the root locus for higher order systems by substituting different values of 'K' in the roots of characteristic equation as used above.

- To simplify the construction of root locus for higher order systems certain rules are developed.

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RULES FOR CONSTRUCTING ROOT LOCUS–

RULE 1:– The root locus is always symmetrical about the real axis. The roots of the characteristic eqⁿ are either real or complex conjugates or combination of both. Therefore, their locus must be symmetrical about the real axis of the s-plane.

RULE 2:– Let $G(s) \cdot H(s)$ = open Loop transfer function of the system and $P$ = No. of open loop poles ; $Z$ = No. of open loop zeros.

i) If $P > Z$ i.e. no. of poles is greater than zeros, then no. of branches equals the no. of open loop poles (i.e. $N = P$)

ii) If $Z > P$ then $N = Z$ [No. of branches equals the no. of O.L zeros]

– for case (i) – Branches will start from each of the location of open loop pole. Out of 'P' no. of branches, 'z' no. of branches will terminate at the locations of open loop zeros and the remaining 'P–Z' branches will approach to infinity.

– for case ii) – Branches will terminate at each of the finite location of open loop zero. But out of 'Z' no. of branches, 'P' no. of branches will start from each of the finite open loop pole locations while remaining 'Z–P' branches will originate from infinity & will approach to finite zeros.

– When $P = Z$, the no. of branches are $N = P = Z$. A separate branch will start from each of the O.L. pole while will terminate at available each O.L. zero. No branch will start or terminate at infinity for $P = Z$.

RULE 3:– Root Loci on the Real Axis

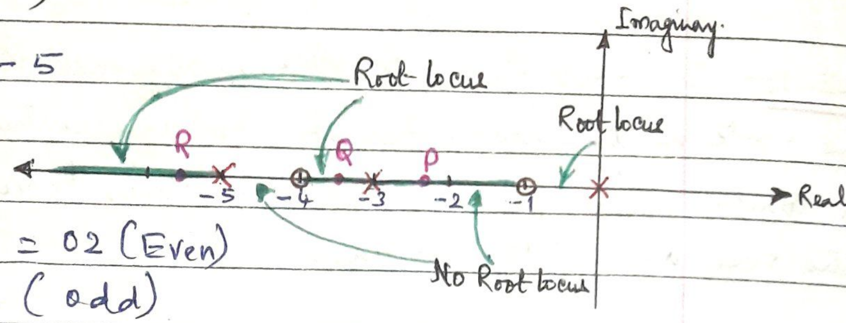
Any point on the real axis is a part of the root locus if and only if the no. of poles and zeros sum is odd to its right.

– Complex poles or zeros are not considered while applying this rule.

Eg:- $G(s)H(s) = \frac{K(s+1)(s+4)}{s(s+3)(s+5)}$, find the sections of real axis where the RL lies.

Soln — Poles — $s = 0, -3, -5$

Zeros — $s = -1, -4$




For pt. P — Sum of poles & zeros to its right $= 1+1 = 02$ (Even)

For Q — $2(P) + 1(Z) = 03$ (odd)

For R — $3(P) + 2(Z) = 05$ (odd)

RULE 4:- Asymptotes —

The branches of root locus tend to infinity along a set of straight line called asymptotes. The total no. of asymptotes $= P - Z$

Angle made by the asymptotes with real axis is given by—

$$\theta = \frac{(2K+1)\,180^\circ}{P-Z}$$

where $K = 0, 1, 2 \ldots (P-Z-1)$

RULE 5:- Centroid of Asymptotes

The point of intersection of asymptotes with real axis is called centroid of asymptotes ($\sigma$) and is given by—

$$\sigma = \frac{\text{Sum of poles} - \text{Sum of Zeros}}{P-Z}$$

or $$\sigma = \frac{\sum \text{Real parts of poles of } G(s)H(s) - \sum \text{Real parts of zeros of } G(s)H(s)}{P-Z}$$

RULE 6:- Breakaway Point —

It is a point on the root locus where multiple roots of the characteristic equation occurs, for a particular value of K.

- Obtain the characteristic equation $1 + G(s) \cdot H(s) = 0$ of the System.
- Obtain the expression of 'K' in terms of 's' i.e. $K = F(s)$
- Differentiate above eqn w.r.t. 's' and equate it to zero.

  i.e. $\frac{dK}{ds} = 0$
- Roots of the eqn $\frac{dK}{ds} = 0$ gives the breakaway point.

- If the value of K is positive, then that breakaway point is valid for the root locus. The breakaway points for which values of 'K' are -ve, those points are invalid.

RULE 7:- Intersection of root locus with imaginary axis—

Steps to be followed-

- Consider the characteristic $eq^n$ obtained in RULE 6.
- Construct Routh's Array in terms of 'K'.
- Determine Kmarginal i.e. the value of K for which one of the rows of Routh's array becomes zero, except the row $s^0$.
- Construct auxiliary $eq^n$ $A(s)=0$ by using coefficients of a row just above the row of zeros.
- Roots of auxiliary $eq^n$ $A(s)=0$ for $K=K_{mar}$ are nothing but the intersection points of the root locus with imaginary axis.

* If $K_{mar}$ is +ve, root locus intersects with imaginary axis else doesn't intersect with imaginary axis and totally lies on the left half of s-plane.

RULE 8:- Angle of Departure at complex pole—

The angle at which a complex conjugate pole departs is known as angle of departure, denoted by $\phi_d$.

where $\phi_d = 180^\circ - \left[\sum\phi_P - \sum\phi_Z\right]$

$\sum\phi_P =$ Contributions by angles made by remaining O.L. poles at the pole where $\phi_d$ is to be calculated.

$\sum\phi_Z =$ Contributions by the angles made by the O.L. zeros at the pole where $\phi_d$ is to be calculated.

RULE 9:- Angle of arrival at complex zero—

$$\phi_a = 180^\circ \left[\sum\phi_Z - \sum\phi_P\right]$$

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Q1. Draw the approximate root locus diagram for the closed loop system whose open loop transfer function is given by -

$$G(s) \cdot H(s) = \frac{K}{s(s+5)(s+10)}$$

. Comment on the stability.

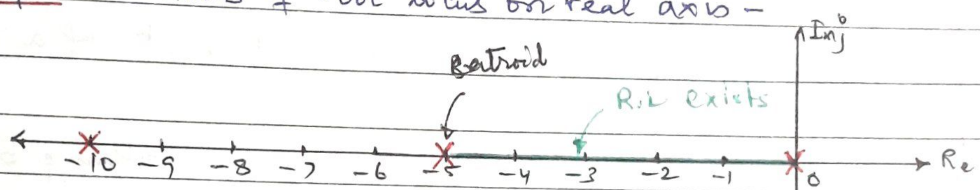
Solⁿ – Step 1 – $P=3$, $Z=0$, $N=P=3$ branches.

$P-Z = 3$ branches approaching $\infty$

Starting points $= 0, -5, -10$

Terminating points $= \infty, \infty, \infty$

Step 2 – Sections of root locus on real axis –




On RHS of Pole at $s=-5$, $P+Z = 1$ (odd)

On RHS of Pole at $s=-10$, $P+Z = 1+1 = 2$ (Even)

Step 3 – Angle of Asymptotes

$$\theta = \frac{(2K+1)180^\circ}{P-Z}, \quad K = 0, 1, 2$$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ, \quad \theta_2 = \frac{3 \times 180^\circ}{3} = 180^\circ, \quad \theta_3 = \frac{5 \times 180^\circ}{3} = 300^\circ$$

Step 4 – Centroid of Asymptotes –

$$\sigma = \frac{\Sigma R.P. \text{ of poles} - \Sigma R.P. \text{ of zeros}}{P-Z} = \frac{0-5-10-0}{3} = -5$$

Step 5 – Breakaway Point

$$1 + G(s)H(s) = 0 \quad \text{or} \quad 1 + \frac{K}{s(s+5)(s+10)} = 0$$

or $s^3 + 15s^2 + 50s + K = 0$ or $K = -s^3 - 15s^2 - 50s$ — (1)

$$\frac{dK}{ds} = -3s^2 - 30s - 50 = 0 \quad \text{or} \quad s^2 + 10s + 16.667 = 0$$

$$\therefore s = \frac{-10 \pm \sqrt{(10)^2 - 4 \times 16.67}}{2} = -2.113, -7.88$$

Substituting $s = -2.113$ in eq^n (1), $K = 48.112$ (valid)

for $s = -7.88$, $K = -48.112$ (invalid)

Step 6 – Intersection with imaginary axis

characteristic eq^n $s^3 + 15s^2 + 50s + K = 0$

| | | |
|---|---|---|
| $s^3$ | 1 | 50 |
| $s^2$ | 15 | K |
| $s^1$ | $\frac{750-K}{15}$ | 0 |
| $s^0$ | K | |

from row of $s^1$, $750 - K = 0$ or $K = 750$ $\therefore K_{max} = 750$

$A(s) = 15s^2 + K = 0$ or $15s^2 + 750 = 0$

or $s = \pm j\sqrt{50} = \pm j7.071$

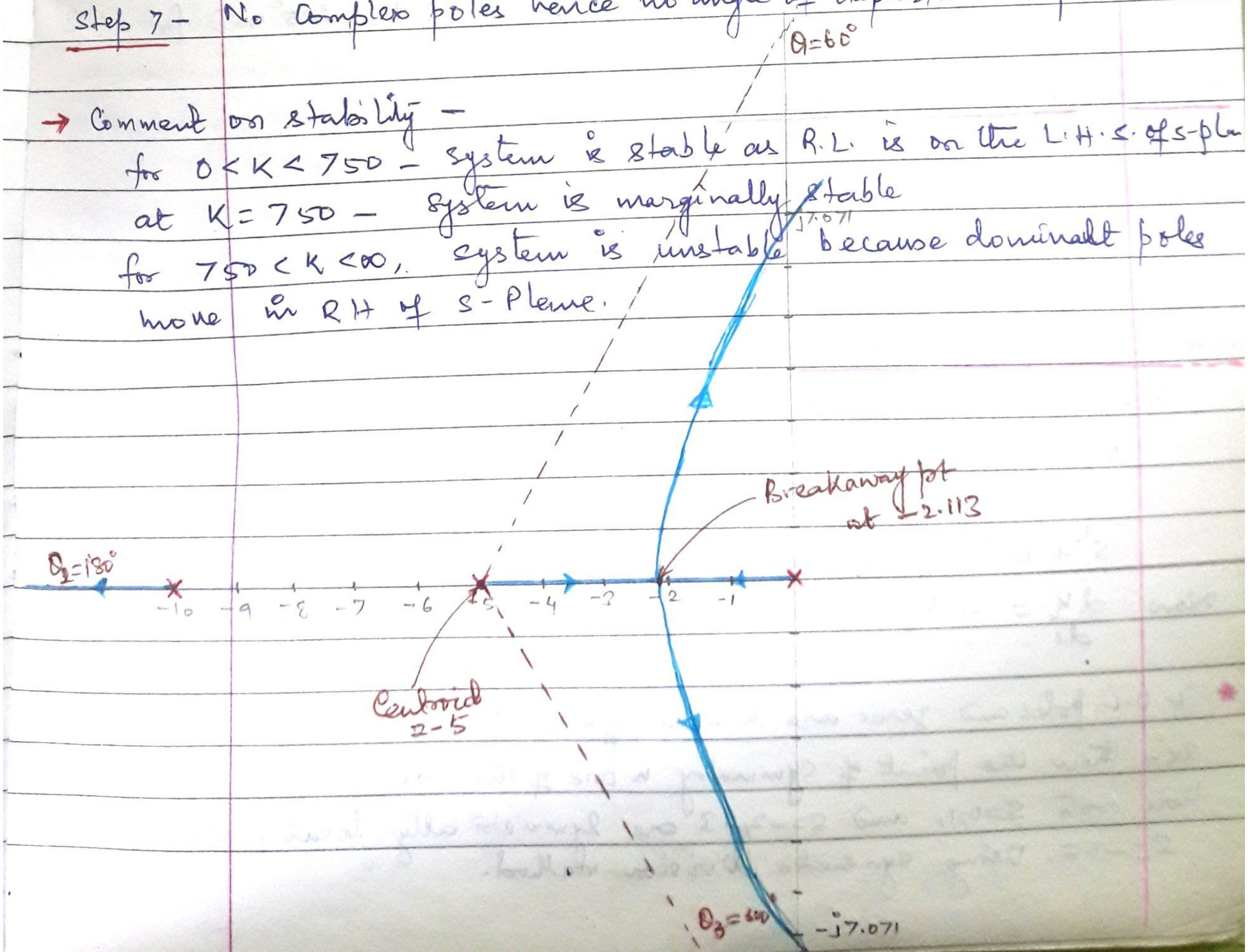
Step 7 – No complex poles hence no angle of departure required.

→ Comment on stability –

for $0 < K < 750$ – system is stable as R.L. is on the L.H.S. of s-plane

at $K = 750$ – system is marginally stable

for $750 < K < \infty$, system is unstable because dominant poles move in RH of s-Plane.



Q2. sketch the complete root locus of the system having O.L.T.F -

$$G(s)\,H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$$

Sol$^n$ – Step 1 – $P=4$, $Z=0$, $N=4$ and all the branches will approach to $\infty$

starting points, $s = 0, -1, -2, -3$

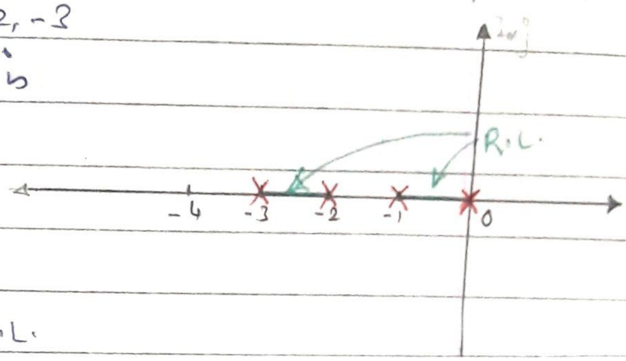
Step 2 – Sections of R.L. on real axis

section b/w 0 and $-1$ → R.L. exists

section b/w $-1$ and $-2$ → No R.L

Section b/w $-2$ and $-3$ → R.L. exists

Section b/w $-3$ and above – No R.L.




Step 3 – Angle of Asymptotes -

$$\theta = \frac{(2K+1)\,180^\circ}{P-Z} \quad \text{where } K = 0, 1, 2, 3$$

for $K=1$, $\theta_1 = 45^\circ$; for $K=2$, $\theta_2 = 135^\circ$; for $K=3$, $\theta_3 = 225^\circ$ and for $K=3$, $\theta_4 = 315^\circ$

Step 4 – Centroid of Asymptotes -

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z}$$

$$\sigma = \frac{0-1-2-3}{4} = \frac{-6}{4} = -1.5$$

Step 5 – Breakaway points -

characteristic Eq$^n$ is $1 + G(s)H(s) = 0$

or $$1 + \frac{K}{s(s+1)(s+2)(s+3)} = 0$$

$s^4 + 6s^3 + 11s^2 + 6s + K = 0$ or $K = -s^4 - 6s^3 - 11s^2 - 6s$

Now $\frac{dK}{ds} = -4s^3 - 18s^2 - 22s - 6 = 0$ or $4s^3 + 18s^2 + 22s + 6 = 0$

* If O.L poles and zeros are located symetorically about a point on the real axis then the point of symmetry is one of the roots of eq$^n$ $\frac{dK}{ds} = 0$

Here roots $s = 0, -1$, and $s = -2, -3$ are symmetorically located about point $s = -1.5$. Using synthetic division method.

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| | | | | |
|---|---|---|---|---|
| −1.5 | 4 | 18 | 22 | 6 |
| | | −6 | −18 | −6 |
| | 4 | 12 | 4 | $\boxed{0}$ |

$\therefore \quad (s+1.5)(4s^2+12s+4) = 0$

Now $\quad 4s^2+12s+4=0 \quad$ or $\quad s^2+3s+1=0$

or $\quad s = \dfrac{-3 \pm \sqrt{9-4\times1}}{2} = \dfrac{-3\pm\sqrt{5}}{2} = -0.381, -2.618$

$\therefore$ Roots of $\dfrac{dK}{ds}=0$ are $s = -1.5, -0.381, -2.618$

* But no root locus exists b/w −1 and −2 hence −1.5 can't be a valid breakaway pt. And for s = −0.381 and −2.618 the value K is +ve, hence both are valid breakaway points.

Step 6 – Intersection with imaginary axis

characteristic eqⁿ – $s^4+6s^3+11s^2+6s+K=0$

Routh Array –

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 11 | K |
| $s^3$ | 6 | 6 | 0 |
| $s^2$ | 10 | K | 0 |
| $s^1$ | $\dfrac{60-6K}{10}$ | 0 | 0 |
| $s^0$ | K | | |

$\therefore \quad \dfrac{60-6K}{10} = 0 \quad \therefore \quad K_{max} = 10$

Auxiliary eqⁿ is $A(s) = 10s^2 + K = 0$, so for $K=10$

$10s^2+10=0 \quad$ or $\quad s^2=-1 \quad$ or $\quad s = \pm j$