

IIR: - have zeros & poles

FIR: - have only zeros.

Discrete Fourier Transform :-

If $x(n)$ is DTS, its N pt DFT is given as -

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N} \cdot n \cdot k}$$

where $k = 0, 1, 2, \dots, N-1$.

and N pt IDFT is given as -

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N} \cdot n \cdot k}$$

where $n = 0, 1, \dots, N-1$

$$e^{-j\frac{2\pi}{N}} = W_N$$

It is called "TWIDDLE FACTOR"

or "PHASE FACTOR"

In terms of twiddle factor.

DFT is -

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk}$$

where $k = 0, 1, \dots, N-1$

and IDFT is -

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}$$

Qo. Determine two pt DFT of sequence

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$$x(n) = \{1, -1\}$$

Here $N=2$.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N} \cdot n \cdot k}$$

and $k = 0, 1$.

$$\therefore X(k) = 1 \cdot e^{-j\frac{2\pi}{2} \cdot 0 \cdot 0} + (-1) e^{-j\frac{2\pi}{2} \cdot 0 \cdot 1}$$

$$= 1 - e^{-j\pi n}$$

$$X(0) = x(0) \cdot e^{-j\pi \cdot 0} + x(1) \cdot e^{-j\pi \cdot 1}$$

$$= 1 + (-1) = 0$$

$$X(0) = 0$$

$$X(1) = x(0) \cdot e^{-j\pi/2 \cdot 0 \cdot 1} + x(1) \cdot e^{-j\pi/2 \cdot 1 \cdot 1}$$

$$= 1 - e^{-j\pi}$$

$$= 1 - (-1) = 2$$

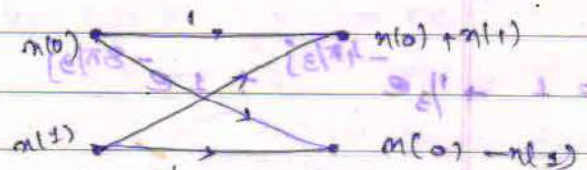
$$X(1) = 2$$

$\therefore X(0) = 0$ for $k=0$.

$X(1) = 2$ for $k=1$.

$$X(k) = \{0, 2\}$$

$$\angle X(k) = \{0, 0\}$$



Two pt DFT

1st sample \Rightarrow sum of $x(0)$ & $x(1)$
2nd sample \Rightarrow diff. b/w $x(0)$ & $x(1)$

Qo. Determine three point DFT

of $x(n) = \left\{ \begin{matrix} 1 \\ 1/3 \\ 1 \end{matrix} \right\}$.

Here, $N=3, k=0,1,2$.

$\therefore X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j \frac{2\pi}{N} \cdot n \cdot k}$

$X(0) = x(0) \cdot e^{-j \frac{2\pi}{3} \cdot 0 \cdot 0} + x(1) \cdot e^{-j \frac{2\pi}{3} \cdot 1 \cdot 0} + x(2) \cdot e^{-j \frac{2\pi}{3} \cdot 2 \cdot 0}$

$= 1 \cdot e^0 + 1/3 e^0 + 1 \cdot e^0$

$X(0) = 2 + 1/3 = 7/3$

$X(1) = x(0) \cdot e^{-j \frac{2\pi}{3} \cdot 1 \cdot 0} + x(1) \cdot e^{-j \frac{2\pi}{3} \cdot 1 \cdot 1} + x(2) \cdot e^{-j \frac{2\pi}{3} \cdot 1 \cdot 2}$

$= 1 \cdot e^0 + 1/3 \cdot e^{-j \frac{2\pi}{3}} + 1 \cdot e^{-j \frac{4\pi}{3}}$

$= 1 + 1/3 [\cos 2\pi/3 - j \sin 2\pi/3]$

$+ 1 [\cos 4\pi/3 - j \sin 4\pi/3]$

$= 1 + 1/3 [-1/2 + -j] + 1 [-1/2 + j]$

$X(1) = 4/3 + \sqrt{3}/3j$

$X(2) = x(0) \cdot e^{-j \frac{2\pi}{3} \cdot 0 \cdot 2} + x(1) \cdot e^{-j \frac{2\pi}{3} \cdot 1 \cdot 2} + x(2) \cdot e^{-j \frac{2\pi}{3} \cdot 2 \cdot 2}$

$= 1 + 1/3 e^{-4\pi/3j} + 1 e^{-8\pi/3j}$

$= 1 + [\cos 4\pi/3 - j \sin 4\pi/3] 1/3 + [\cos 8\pi/3 - j \sin 8\pi/3]$

$+ [\cos 0 - j \sin 0]$

$X(2) = 1 + [-1/2 + j \frac{\sqrt{3}}{2}] + [-1/2 + j \frac{\sqrt{3}}{2}]$

Discrete Fourier Transform

$= 1 - 1/6 - \frac{\sqrt{3}}{3 \cdot 2}j - 1/2 + \frac{\sqrt{3}}{2}j$

$= \frac{6-1-3}{6} - \frac{\sqrt{3}j}{2} [1/3 - 1]$

$= \frac{2}{6} - \frac{\sqrt{3}j}{2} [\frac{-2}{3}]$

$X(2) = 1/3 - 1/3j$

$\therefore X(0) = 7/3$

$X(1) = 4/3 + j/\sqrt{3}$

$X(2) = 1/3 - 1/\sqrt{3}j$

Three point DFT of $x(n) = \{1, 1/3, 1\}$

$X(0) = 7/3 \angle 0^\circ$

$X(1) = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} \angle \tan^{-1} \left(\frac{1/\sqrt{3}}{4/3}\right)$

$X(2) = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} \angle \tan^{-1} \left(\frac{-1/\sqrt{3}}{1/3}\right)$

Qo. Determine four point DFT

of $x(n) = \{x(0), x(1), x(2), x(3)\}$

$x(n) = \cos n\pi/4, n=0,1,2,3$

$\therefore x(n) = \left\{ 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\}$

$$x(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j \frac{2\pi}{N} \cdot n \cdot k}$$

$$N=3, \quad k=0, 1, 2, 3$$

$$x(0) = x(0) \cdot e^{-j \frac{2\pi}{4} \cdot 0 \cdot 0} +$$

$$x(1) \cdot e^{-j \frac{2\pi}{4} \cdot 1 \cdot 0} + x(2) \cdot e^{-j \frac{2\pi}{4} \cdot 2 \cdot 0}$$

$$+ x(3) \cdot e^{-j \frac{2\pi}{4} \cdot 3 \cdot 0}$$

$$= 1 \cdot e^0 + \frac{1}{\sqrt{2}} \cdot e^0 + 0 - \frac{1}{\sqrt{2}} \cdot e^0$$

$$x(0) = 1$$

$$x(1) = x(0) \cdot e^{-j \frac{2\pi}{4} \cdot 1 \cdot 0} + x(1) \cdot e^{-j \frac{2\pi}{4} \cdot 1 \cdot 1}$$

$$+ x(2) \cdot e^{-j \frac{2\pi}{4} \cdot 2 \cdot 1} + x(3) \cdot e^{-j \frac{2\pi}{4} \cdot 3 \cdot 1}$$

$$= 1 + \frac{1}{\sqrt{2}} \cdot e^{-\pi/2j} + 0 - \frac{1}{\sqrt{2}} \cdot e^{-3\pi/4j}$$

$$= 1 + \frac{1}{\sqrt{2}} \left[\cos \frac{\pi}{2} - j \sin \frac{\pi}{2} \right] - \frac{1}{\sqrt{2}} \left[\cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2} \right]$$

$$= 1 + \frac{1}{\sqrt{2}} [0 - j] - \frac{1}{\sqrt{2}} [0 + j]$$

$$= 1 - \frac{1}{\sqrt{2}} j - \frac{1}{\sqrt{2}} j = 1 - \sqrt{2} j$$

$$x(1) = 1 - \sqrt{2} j$$

$$x(2) = x(0) \cdot e^{-j \frac{2\pi}{4} \cdot 2 \cdot 0} + x(1) \cdot e^{-j \frac{2\pi}{4} \cdot 2 \cdot 1}$$

$$+ x(2) \cdot e^{-j \frac{2\pi}{4} \cdot 2 \cdot 2} + x(3) \cdot e^{-j \frac{2\pi}{4} \cdot 2 \cdot 3}$$

$$= 1 + \frac{1}{\sqrt{2}} e^{-\pi j} + \frac{1}{\sqrt{2}} e^{-3\pi j}$$

$$= 1 + \frac{1}{\sqrt{2}} \left[\cos \pi - j \sin \pi \right] - \frac{1}{\sqrt{2}} \left[\cos 3\pi - j \sin 3\pi \right]$$

$$= 1 + \frac{1}{\sqrt{2}} [-1 - 0] - \frac{1}{\sqrt{2}} [-1 - 0]$$

$$x(2) = 1$$

$$x(3) = x(0) \cdot e^{-j \frac{2\pi}{4} \cdot 3 \cdot 0} +$$

$$x(1) \cdot e^{-j \frac{2\pi}{4} \cdot 3 \cdot 1} +$$

$$x(2) \cdot e^{-j \frac{2\pi}{4} \cdot 3 \cdot 2} + x(3) \cdot e^{-j \frac{2\pi}{4} \cdot 3 \cdot 3}$$

$$= 1 + \frac{1}{\sqrt{2}} \cdot e^{-3/2\pi j} - \frac{1}{\sqrt{2}} \cdot e^{-9\pi/2j}$$

$$= 1 + \frac{1}{\sqrt{2}} \left[\cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2} \right]$$

$$- \frac{1}{\sqrt{2}} \left[\cos \frac{9\pi}{2} - j \sin \frac{9\pi}{2} \right]$$

$$x(2) = 1 + \sqrt{2} j$$

$$\therefore x(k) = \{ 1, 1 - \sqrt{2} j, 1, 1 + \sqrt{2} j \}$$

Deduction -

in terms of twiddle factor.

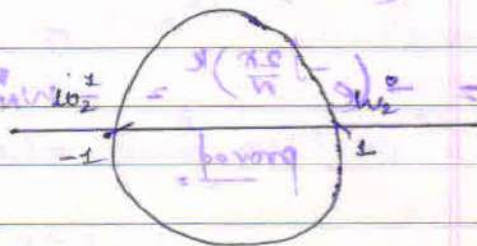
$$x(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{n \cdot k}$$

$$x(0) = x(0) + x(1) \cdot \dots + x(3)$$

$$x(1) = x(0) + x(1) \cdot W^1 + x(2) \cdot W^2 + x(3) \cdot W^3$$

$$x(2) = x(0) + x(1) \cdot W^2 + x(2) \cdot W^4 + x(3) \cdot W^6$$

$$W_N = 1 \cdot e^{-j \frac{2\pi}{N}} = r e^{j\theta}$$



• properties of Twiddle factor :-

1. Periodicity :-

It is always periodic fn.

$$W_N^{(k+N)} = W_N^k.$$

$$W_N^{(k+rN)} = W_N^k.$$

proof :- let $W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore \left(e^{-j\frac{2\pi}{N}} \right)^{(k+rN)} = e^{-j\frac{2\pi}{N} \cdot k} \cdot e^{-j\frac{2\pi}{N} \cdot rN}$$

$$= e^{-j\frac{2\pi}{N} \cdot k} \cdot \left(e^{-j2\pi} \right)^r$$

$$= e^{-j2\pi/N \cdot k} = W_N^k.$$

proved.

2. Half Periodicity :-

$$W_N^{(k+N/2)} = -W_N^k.$$

proof :- $W_N = e^{-j\frac{2\pi}{N}}$

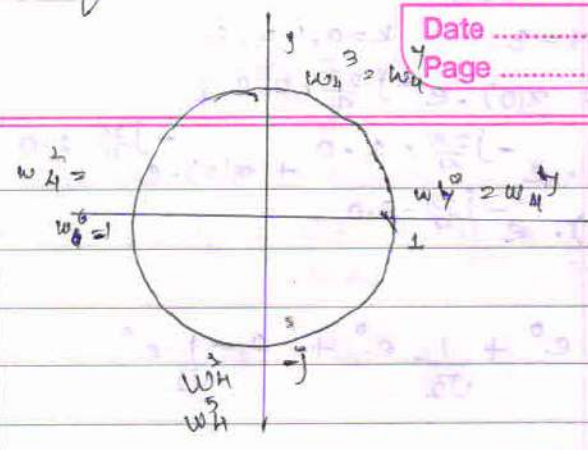
$$\therefore \left(e^{-j\frac{2\pi}{N}} \right)^{(k+N/2)} = \left(e^{-j\frac{2\pi}{N}} \right)^k \cdot \left(e^{-j\frac{2\pi}{N}} \right)^{N/2}$$

$$= \left(e^{-j\frac{2\pi}{N}} \right)^k \cdot (-1)$$

$$= - \left(e^{-j\frac{2\pi}{N}} \right)^k = -W_N^k.$$

proved.

for $N=4$.



for half periodicity

$$W_4^3 = W_4^{(1+4/2)} = -W_4^1$$

$$W_4^2 = W_4^{(0+4/2)} = -W_4^0$$

$$3. W_N^N = 1, W_N^{N/2} = -1$$

Notes - Shortcut method

$$\text{let } x(n) = \left\{ 1, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\}$$

$$X(k) = \sum_{n=0}^3 x(n) \cdot W_4^{nk}$$

$$X(0) = x(0) + x(1) + x(2) + x(3) = 1$$

$$X(1) = \sum_{n=0}^3 x(n) \cdot W_4^n$$

$$= x(0) + x(1)W_4^{-1} + x(2)W_4^2 + x(3)W_4^3$$

$$= 1 + \frac{1}{\sqrt{2}}(-1) + 0(-1) + \frac{1}{\sqrt{2}}$$

$$X(2) = x(0) + x(1) \cdot w_4^2 + x(2) w_4^4 + x(3) \cdot w_4^6$$

$$= 1 + \frac{1}{\sqrt{2}}(-1) + 0 + \left(\frac{-1}{\sqrt{2}}\right)(-1)$$

$$= 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1$$

$$X(3) = x(0) + x(1) w_4^3 + x(2) w_4^6 + x(3) \cdot w_4^9$$

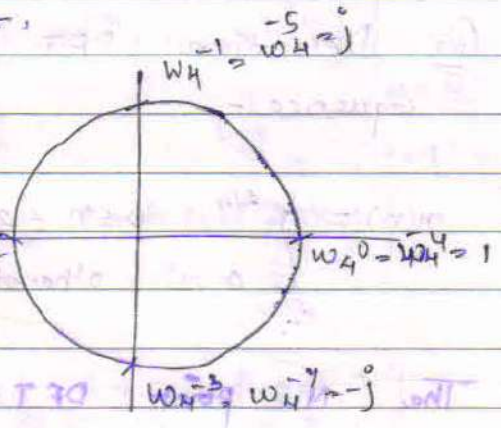
$$= 1 + \frac{1}{\sqrt{2}}(j) + 0 + \left(\frac{-1}{\sqrt{2}}\right)(-j)$$

$$= 1 + \frac{j}{\sqrt{2}} + \frac{j}{\sqrt{2}} = 1 + \sqrt{2}j$$

$$X(K) = \{ 1, 1 - \sqrt{2}j, 1, 1 + \sqrt{2}j \}$$

Note: For DFT → Clockwise
IDFT → anticlockwise

for IDFT



$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot w_N^{-nk}$$

$$n = 0, 1, 2, \dots$$

$$x(n) = \frac{1}{4} \sum_{k=0}^3 X(k) \cdot w_4^{-nk}$$

$$x(0) = \frac{1}{4} \sum_{k=0}^3 x(k) \cdot w_4^{0k}$$

$$= \frac{1}{4} [x(0) + x(1) + x(2) + x(3)]$$

$$= \frac{1}{4} [1 + (1 - \sqrt{2}j) + 1 + (1 + \sqrt{2}j)]$$

$$= \frac{8}{4} = 2$$

$$x(1) = \frac{1}{4} \sum_{k=0}^3 x(k) \cdot w_4^{-k}$$

$$= \frac{1}{4} [x(0) w_4^{-0} + x(1) w_4^{-1} + x(2) w_4^{-2} + x(3) w_4^{-3}]$$

$$= \frac{1}{4} [1 \cdot 1 + (1 - \sqrt{2}j)j + 1 \cdot (-1) + (1 + \sqrt{2}j) \cdot (-j)]$$

$$= \frac{1}{4} [1 + j + \sqrt{2} - 1 - j + \sqrt{2}]$$

$$= \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$x(2) = \frac{1}{4} \sum_{k=0}^3 x(k) \cdot w_4^{2k}$$

$$= \frac{1}{4} [x(0) \cdot w_4^{-0} + x(1) \cdot w_4^{-2} + x(2) \cdot w_4^{-4} + x(3) \cdot w_4^{-6}]$$

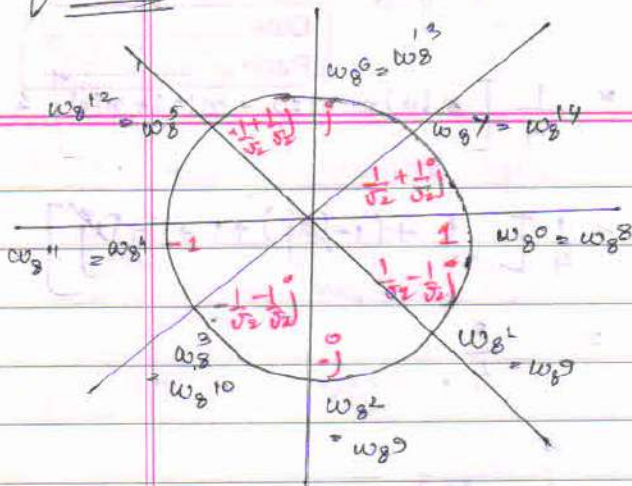
$$= \frac{1}{4} [1 \cdot (1) + (1 + \sqrt{2}j)(-1) + 1 \cdot (1) + (1 - \sqrt{2}j)(-1)]$$

$$x(2) = 0$$

$$x(3) = \frac{1}{4} \sum_{k=0}^3 x(k) \cdot w_4^{-3k}$$

$$x(3) = \frac{1}{\sqrt{2}}$$

for $N=8$



$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^1 & \dots & W_N^{N-1} \\ 1 & W_N^2 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \dots & \vdots \\ 1 & W_N^{N-1} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$N \times 1$ $N \times N$ $N \times 1$

\rightarrow DFT and IDFT are linear transformation

Ex:-

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix}$$

Since, DFT \Rightarrow

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk}$$

$k = 0, 1, \dots, N-1$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$x(0) = \sum_{n=0}^{N-1} x(n) = x(0) + x(1) + x(2) + \dots$$

Qo. Determine DFT of the sequence -

$$x(1) = x(0) + x(1) \cdot W_N^1 + x(2) \cdot W_N^2 + \dots + x(N-1) \cdot W_N^{N-1}$$

$$x(n) = \begin{cases} 1/4 & 0 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$x(N-1) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{(N-1)n}$$

The $(N-1)$ th point DFT of $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{j \frac{2\pi}{N} nk}$$

$k = 0, 1, \dots, (N-1)$

This can be expressed as matrix.

$$x(n) = (1/4, 1/4, 1/4)$$

$$\therefore x(k) = \frac{1}{4} \left[1 + e^{-j\omega} + e^{j2\omega} \right]$$

$$\omega = \frac{2\pi k}{N}$$

$$= \frac{1}{4} e^{-j2\pi k/3} \left[1 + 2 \cos\left(\frac{2\pi k}{3}\right) \right]$$

Hence,

$$x(k) = \frac{1}{4} e^{-j\frac{2\pi k}{3}} \left[1 + 2 \cos\left(\frac{2\pi k}{3}\right) \right]$$

where $k = 0, 1, \dots, N-1$

Qo. Determine DFT of sequence

$$x(n) = \begin{cases} 1/5 & -1 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right)$$

$$X(K) = \sum_{n=0}^{\infty} x(n) e^{-j\omega n} \text{ at } \omega = \frac{2\pi k}{N}$$

$$X(K) = \frac{1}{5} \left[e^{j\omega} + 1 + e^{-j\omega} \right]$$

$$\omega = \frac{2\pi k}{N}$$

$$= \frac{1}{5} \left[1 + 2 \cos\left(\frac{2\pi k}{3}\right) \right]$$

$$X(K) = \frac{1}{5} \left[1 + 2 \cos\left(\frac{2\pi k}{3}\right) \right]$$

→ ans

Qo Find N-point DFT for

$$x(n) = a^n \quad 0 < a < 1$$

$$X(K) = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}nk} \quad k = 0, 1, \dots, (N-1)$$

$$X(K) = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi nk}{N}}$$

$$= 1 - (ae^{-j\frac{2\pi k}{N}})^N$$

$$1 - ae^{-j2\pi k/N}$$

$$X(K) = \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

Qo. Determine IDFT of

$$X(K) = \{3, (2+j), 1, (2-j)\}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) e^{j\frac{2\pi nk}{N}}$$

$$N=4, \quad x(n) = \frac{1}{4} \sum_{k=0}^3 X(K) e^{j\frac{2\pi nk}{4}}$$

when $n=0$

$$x(0) = \frac{1}{4} \sum_{k=0}^3 X(K) e^0 = \frac{1}{4} [3 + (2+j) + 1 + (2-j)] = 2$$

When $n=1$

$$x(1) = \frac{1}{4} \left[3 + (2+j) e^{j\pi/2} + e^{j\pi} + (2-j) e^{j3\pi/2} \right]$$

$$= \frac{1}{4} [3 + (2+j)j - 1 + (2-j)(-j)]$$

$$= 0$$

when $n=2$

$$x(2) = \frac{1}{4} [3 + (2-j)(-1) + 1 + (2-j)(-1)]$$

$$(2-j)(-1) = 0$$

When $n=3$

$$n(s) = \frac{1}{4} \sum_{k=0}^3 X(k) e^{j3\pi k/2}$$

$$n(3) = \frac{1}{4} [3 + (2+j)(-j) - 1$$

$$+ (2-j)(j)] = 1$$

$\therefore a(n) = \{2, 0, 0, 1\} \rightarrow \underline{am}$

~~Imp~~

properties of DFT :-

1. Linearity
2. Periodicity
3. Circular shift of a sequence
4. Time Reversal
5. Circular time shift
6. Circular freq. shift
7. Circular Convolution
8. Complex Conjugation
9. Multiplication of two sequences
10. Parseval's theorem
11. Symmetric property

Linearity :-

$$x_1(n) \xrightarrow{\text{DFT}} X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{nk}$$

$$k = 0, 1, \dots, N-1$$

$$x_2(n) \xrightarrow{\text{DFT}} X_2(k) = \sum_{n=0}^{N-1} x_2(n) W_N^{nk}$$

$$k = 0, 1, \dots, (N-1)$$

then,

$$a = [(a-j)(j) - 1]$$

$$x_3(k) = a_1 x_1(k) +$$

$$a_2 x_2(k)$$

$$k = 0, 1, \dots, (N-1)$$

proof :-

$$\text{DFT}[x_3(n)] = \sum_{n=0}^{(N-1)} x_3(n) W_N^{nk}$$

$$k = 0, 1, \dots, (N-1)$$

$$= \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] W_N^{nk}$$

$$k = 0, 1, \dots, (N-1)$$

$$= a_1 X_1(k) + a_2 X_2(k)$$

proved

2. Periodicity :-

$$x(n) \xrightarrow{\text{DFT}} X(k)$$

then, $X(k+N) = X(k)$

$$\text{proof: } X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{n(k+N)}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{nk} \cdot W_N^{nN}$$

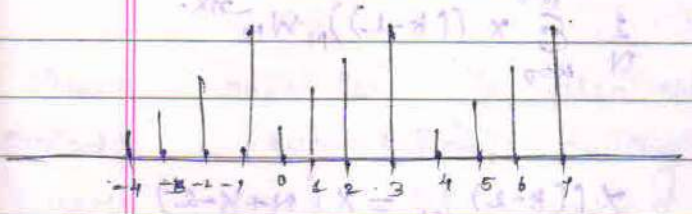
$$\text{Here } W_N^{nN} = \left(e^{-j\frac{2\pi}{N}} \right)^{nN} = (e^{-j2\pi})^n = 1$$

$$\therefore X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = X(k)$$

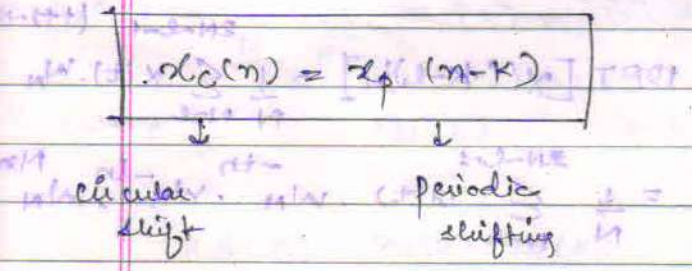
proved

3. Circular shift of a Sequence :-
 In DFT we cannot combine linear shifted seq. with the original one due to loss of signal in the period. So we do periodic shifting.

original one due to loss of signal in the period. So we do periodic shifting.



For periodic shifting - sequences can be (1 2 3 4) (4 1 2 3) (3 4 1 2) (2 3 4 1) (1 2 3 4)



$k =$ no. of positions (shifted/shifting) periods.

- ie $x_c(0) = x_p(0-1) = 4$
- $x_c(1) = x_p(1-1) = 1$
- $x_c(2) = x_p(2-1) = 2$
- $x_c(3) = x_p(3-1) = 3$

another way of representing periodic shifting.
 $N =$ periodicity.

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Exo. $k=1$ $N=4$
 then.
 $x_c(n) = x((N+n-k))$
 $x_c(0) = x(4+0-1) = x(3)$
 $x_c(1) = x(4+1-1) = x(4)$
 $x_c(2) = x(4+2-1) = x(1)$
 $x_c(3) = x(4+3-1) = x(2)$

4. Time Reversal :-

If $x(n) \xrightarrow{\text{DFT}} X(k)$

then,
 $x((-n))_N \xrightarrow{\text{DFT}} X((-k))_N$

$x(N-n) \xrightarrow{\text{DFT}} X(N-k)$

proof :- $\text{DFT} [x((-n))] = \sum_{n=0}^{N-1} x((-n)) W_N^{nk}$

$k=0, 1, \dots, (N-1)$

$x((-n))_N = x(N-n)$
 $= \sum_{n=0}^{N-1} x(N-n) W_N^{nk}$

$k=0, 1, \dots, (N-1)$

Let $N-n=t$.

$\text{DFT} [x((-n))_N] = \sum_{t=N}^1 x(t) \cdot W_N^{(N-t)k}$
 $= \sum_{t=N}^1 x(t) \cdot W_N^{-tk}$

$\text{DFT} [x((-n))_N] = \left(\sum_{t=0}^{N-1} x(t) \right) W_N^{-tk}$

$$\text{DFT} [x((n-l))_N] = \sum_{t=0}^{N-1} x(t) W_N^{-tk} = \sum_{t=0}^{N-1} x(t) W_N^{-tk}$$

$$= \sum_{t=N-l}^{N-1} x(t) W_N^{-tk} = \sum_{t=0}^{N-1} x(t) W_N^{-(N-k)t} = x((N-k))_N$$

5. Circular time shift

$$x(n) \xrightarrow[\text{DFT}]{N} X(K)$$

then,

$$x((n-l)) \xrightarrow[\text{DFT}]{N} W_N^{lk} X(K)$$

proof

$$\text{DFT} [x((n-l))_N] = \sum_{t=0}^{N-1} x(t) W_N^{-tk} = \sum_{t=0}^{N-1} x(t) W_N^{-tk}$$

put $N+n-l = t$

then

$$\text{DFT} [x((n-l))_N] = \sum_{t=0}^{N-1} x(t) W_N^{-tk}$$

$$= \sum_{t=0}^{N-1} x(t) W_N^{-tk} = \sum_{t=0}^{N-1} x(t) W_N^{-tk}$$

$$= \sum_{t=0}^{N-1} x(t) W_N^{-tk} = \sum_{t=0}^{N-1} x(t) W_N^{-tk}$$

$$= W_N^{lk} X(K) = \text{DFT} [x((n-l))_N]$$

proved

6. Circular freq. shift

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if $x(n) \leftrightarrow X(K)$
then,

$$x(n) W_N^{-ln} \xrightarrow[\text{DFT}]{N} X((K-l))_N$$

proof:

$$\text{IDFT} [X((K-l))_N] =$$

$$\frac{1}{N} \sum_{k=0}^{N-1} X((K-l))_N W_N^{-nk}$$

$$X((K-l))_N = X((N+K-l))$$

$$\text{IDFT} [X((K-l))_N] = \frac{1}{N} \sum_{k=0}^{N-1} X((N+K-l)) W_N^{-nk}$$

put

$$\text{IDFT} [X((K-l))_N] = \frac{1}{N} \sum_{t=0}^{N-1} x(t) W_N^{-nt} = \frac{1}{N} \sum_{t=0}^{N-1} x(t) W_N^{-nt}$$

$$\text{IDFT} [X((K-l))_N] = W_N^{-ln} \frac{1}{N} \sum_{t=0}^{N-1} x(t) W_N^{-nt}$$

$$= W_N^{-ln} X(K) = \text{DFT} [x(n) W_N^{-ln}]$$

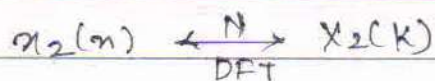
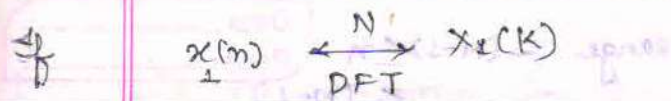
now taking DFT on both sides we get -

$$X((K-l))_N = \text{DFT} [W_N^{-ln} x(n)]$$

$$\therefore X((K-l))_N = \text{DFT} [W_N^{-ln} x(n)]$$

proved

4. Circular Convolution :-



and $X_3(K) = X_1(K) \cdot X_2(K)$

then,

$x_3(n) = x_1(n) \odot x_2(n)$

circular convolution is also called periodic convolution, having length equal to length of convolved sequences.

Proof :- $x_3(n) = \text{IDFT} [X_3(K)]$

$x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(K) W_N^{-nk}$

$n = 0, 1, \dots, (N-1)$

Since,

$X_3(K) = X_1(K) \cdot X_2(K)$

$\therefore x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(K) \cdot X_2(K) \cdot W_N^{-nk}$ (i)

let

$X_1(K) = \sum_{m=0}^{N-1} x_1(m) W_N^{mK}$ (ii)

$X_2(K) = \sum_{l=0}^{N-1} x_2(l) W_N^{lK}$ (iii)

$K = 0, 1, \dots, (N-1)$

Now substituting the values from eq. (ii) & (iii) into eq. (i),

$\therefore x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m) W_N^{mK} \right] \cdot \left[\sum_{l=0}^{N-1} x_2(l) W_N^{lK} \right]$

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$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \cdot \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} W_N^{(m+l-k)k}$

$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \cdot \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} W_N^{(m+l-k)k}$

for $\sum_{k=0}^{N-1} W_N^{(m+l-k)k}$

suppose $W_N^{(m+l-k)} = a$

$\therefore \sum_{k=0}^{N-1} a^k = \begin{cases} \frac{1-a^N}{1-a} & \text{when } a \neq 1 \\ N & a = 1 \end{cases}$

Here, we see that considering $\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a}$

then eq. (iv) becomes zero.

\therefore we will consider the next case. i.e. $\sum_{k=0}^{N-1} a^k = N, (a=1)$

$\frac{1 - W_N^{(m+l-n)N}}{1 - W_N^{(m+l-n)}}$

for existence of circular conv.

$a = 1$

i.e. $W_N^{(m+l-n)} = 1$

when,

$m+l-n = qN$

$x_3(n) = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l)$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \delta[n - (m+l)]$$

$$m+l-n = qN$$

$$l = qN + n - m$$

$$l = ((n-m))_N$$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N$$

(circular conv.)

$$n = 0, 1, \dots, (N-1)$$

$$x_3(n) = \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

$$= x_1(n) \otimes x_2(n)$$

(linear conv.)

$$x_3(n) = x_1(n) \odot x_2(n)$$

proved

Methods of Circular Convolution :-

- Tabular
- Using Circular array
- Using DFT & IDFT (Mistochastic Method)
- Matrix

Q. Determine \odot of $x_1(n) = \{1, 2, 2, 1\}$

$$x_2(n) = \{2, 1, 1, 2\}$$

By Tabular Method -

range $-(N-1) \leq n \leq (N-1)$

	-3	-2	-1	0	1	2	3
$x_1(n)$				1	2	2	1
$x_2(n)$				2	1	1	
$x_{lp}(n)$	1	1	2	2	1	1	
$x_{hp}(n)$	2	1	1	2	2	1	
$x_3(n)$		2	1	1	2	2	
$x_3(2-n)$			2	1	1	2	
$x_3(n)$				2	1	1	

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) x_2(n-m)$$

$$n = 0, 1, \dots, (N-1)$$

$$n = 0$$

$$x_3(0) = 1 \times 2 + 2 \times 2 + 2 \times 1 + 1 \times 1 = 9$$

$$x_3(1) = 1 \times 1 + 2 \times 2 + 2 \times 1 + 1 \times 2 = 10$$

$$x_3(2) = 1 \times 1 + 2 \times 1 + 2 \times 2 + 1 \times 2 = 9$$

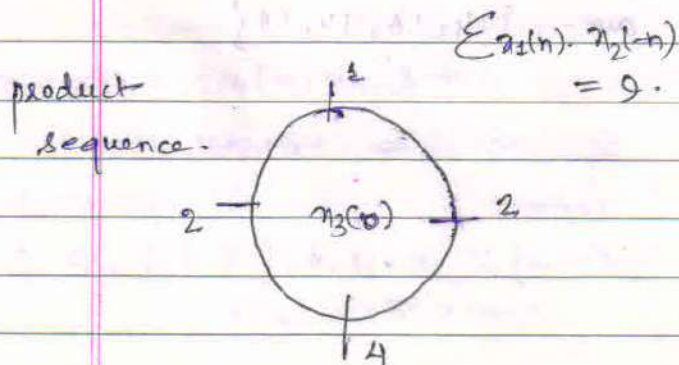
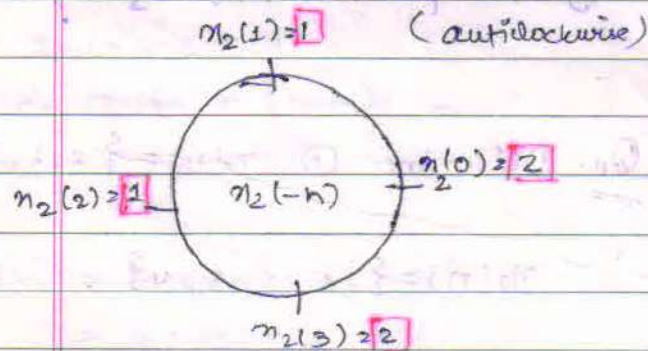
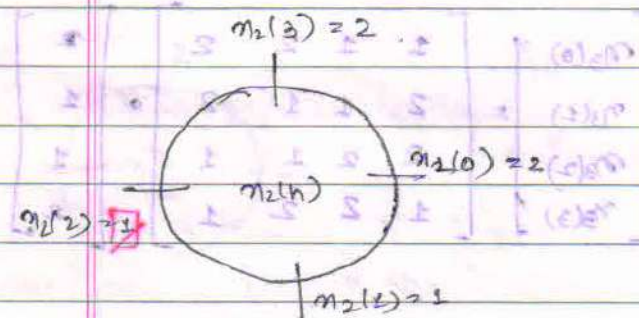
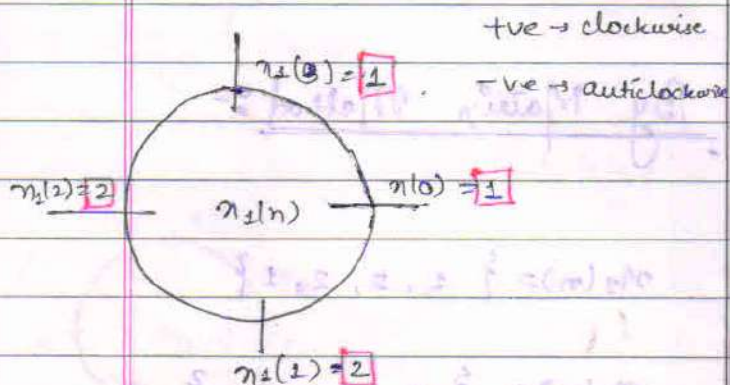
$$x_3(n) = 1 \times 2 + 2 \times 1 + 2 \times 1 + 1 \times 2 = 8$$

$x_3(n) = \{9, 10, 9, 8\}$

By Using Circular Array :-

$x_1(n) = \{1, 2, 2, 1\}$

$x_2(n) = \{2, 1, 1, 2\}$

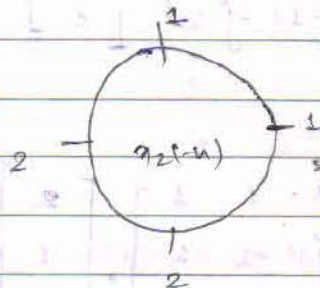


$x_3(n) = x_1(n) +$

clockwise shift of $x_2(-n)$.

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$\sum x_1(n) \cdot x_2(-n)$

$x_3(1) = 1 \times 1 + 2 \times 2 + 2 \times 2 + 1 \times 1 = 10$

$x_3(2) = x_1(n) + 2$ times clockwise shift of $x_2(-n)$

$x_3(2) = 2 \times 1 + 1 \times 1 + 1 \times 2 + 2 \times 2 = 9$

$x_3(3) = x_1(n) + 3$ times clockwise shift of $x_2(-n)$

$x_3(n) = 0$

$x_3(n) = \{9, 10, 9, 8\}$

By Using DFT & IDFT Method :-

$x_3(k) = x_1(k) \cdot x_2(k)$

$x_3(n) = x_1(n) \odot x_2(n)$

$x_1(n) = \{1, 2, 2, 1\}$

$x_2(n) = \{2, 1, 1, 2\}$

$$\begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -1 \\ 1 & -j & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_2(0) \\ x_2(1) \\ x_2(2) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & 1 \\ 1 & -j & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$x_3(n) = \{ 9, 10, 9, 8 \}$$

By Matrix Method:-

$$\begin{aligned} x_1(0) &= 1 + 2 + 2 + 1 = 6 \\ x_1(1) &= 1 - 2j - 2 + j = -1 - j \\ x_1(2) &= 1 - 2 + 2 - j = 1 - j \\ x_1(3) &= 2 + j - 1 - 2j = 1 - j \end{aligned}$$

$$x_4(n) = \{ 1, 2, 2, 1 \}$$

$$x_2(n) = \{ 2, 1, 1, 2 \}$$

$$\begin{aligned} x_2(0) &= 6 & x_2(1) &= 1 + j \\ x_2(2) &= 0 & x_2(3) &= 1 - j \end{aligned}$$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$x_1(k) = \{ 6, (-1-j), 0, (-1+j) \}$$

$$x_2(k) = \{ 6, (1+j), 0, (1-j) \}$$

$$x_3(n) = \{ 9, 10, 9, 8 \}$$

$$x_3(k) = \{ 36, -2j, 0, 2j \}$$

Qo. Determine (i) $x_4(n) = \{ 2, 1, 2 \}$

$$x_2(n) = \{ 1, 2, 3, 4 \}$$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ 1 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ 1 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} \begin{bmatrix} 36 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

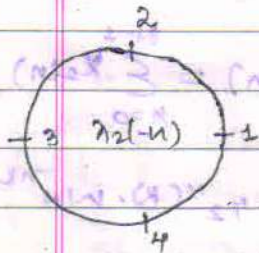
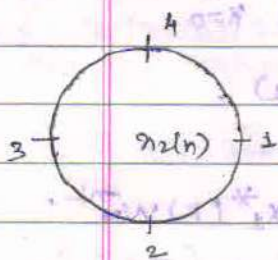
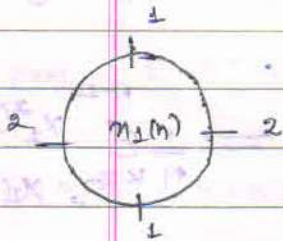
$$\text{ans} = \{ 14, 16, 14, 16 \}$$



Circular array.

$$n_1(n) = \{2, 1, 2, 1\}$$

$$n_2(n) = \{1, 2, 3, 4\}$$



$$n_3(0) = \sum n_1(m) n_2(-m)$$

$$= 2 + 6 + 4 + 2 = 14$$

$$n_3(1) = \sum n_1(m) n_2(1-m)$$

$$= 1 + 8 + 3 + 4 = 16$$

$$n_3(2) = \sum n_1(m) n_2(2-m)$$

$$= 6 + 4 + 2 + 2 = 14$$

$$n_3(3) = \sum n_1(m) n_2(3-m)$$

$$= 4 + 3 + 8 + 1 = 16$$

$$\therefore n_3(n) = \{14, 16, 14, 16\}$$

Matrix Method.

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$$n_3(n) = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} =$$

$$n_3(n) = \{14, 16, 14, 16\}$$

3. Complex Conjugate Property :-

$$\text{If } n(n) \xleftrightarrow{\text{DFT}} x(k)$$

$$\text{then } n^*(n) \xleftrightarrow{\text{DFT}} x^*((-k))_N \\ = x^*(N-k)$$

$$\text{proof :- DFT}[n(n)] = \sum_{n=0}^{N-1} n(n) W_N^{nk}$$

$$k = 0, 1, \dots, (N-1)$$

$$\text{DFT}[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{nk}$$

$$k = 0, 1, \dots, (N-1)$$

$$\text{DFT}[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{-nk} = W_N^{-nk}$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-k)n}$$

(multiplying $W_N^{-Nn} = 1$)

$$\text{DFT}[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{nk} W_N^{-Nn}$$

$$= \sum_{n=0}^{N-1} [x(n) W_N^{(N-k)n}]^*$$

$$= x^*(N-k)$$