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NUMERICAL ANALYSIS – III.

Numerical solution of Ordinary Differential Equation

Consider the diff. eq^n of first order and first degree. $\frac{dy}{dx} = f(x,y)$

with condition $y(x_0) = y_0$

Methods.
1. Picard's Method.
2. Taylor's Method
3. Euler Method
4. Euler- Modified Method
5. Runge's Kutta Method
6. Milne's Predictor- Corrector Method
7. Adam's Bashforth Method.

1. Picard's Method.

Consider the diff. eq^n

$$\frac{dy}{dx} = f(x,y) \quad \text{—— (1)}.$$

with condition $Y(x_0) = Y_0$.

Integrate both side by eq^n (1)

b/w the limits

$$\int_{y_0}^{y} \left(\frac{dy}{dx}\right) dx = \int_{x_0}^{x} f(x,y)\, dx$$

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$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\boxed{y = y_0 + \int_{x_0}^{x} f(x,y)\,dx}$$

First approx.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

2nd. $$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

3rd. $$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$$

$$\vdots$$

$$\boxed{y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx}$$

Q. Use Picard Method to obtain y for $x = 0.2$ given $\frac{dy}{dx} = x - y$ with Initial condition $y = 1$ when $x = 0$

Given $f(x,y) = x - y$

$x_0 = 0, \quad y_0 = 1$

1st approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_{0}^{x} (x - y_0)\,dx$$

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$$= 1 + \int_0^x (x-1)\,dx$$

$$= 1 + \left[\frac{x^2}{2} - x\right]_0^x$$

$$= 1 + \left(\frac{x^2}{2} - x\right)$$

$$y^{(1)} = 1 - x + \frac{x^2}{2}$$

$$y^{(1)}(0.2) = 0.82$$

2nd.

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$$= 1 + \int_0^x [x - y^{(1)}]\,dx$$

$$= 1 + \int_0^x x - \left(1 - x + \frac{x^2}{2}\right) dx$$

$$= 1 + \int_0^x x - 1 + x - \frac{x^2}{2}\;dx$$

$$= 1 + \int_0^x \left(2x - 1 - \frac{x^2}{2}\right) dx$$

$$= 1 + \left[x^2 - x - \frac{x^3}{6}\right]_0^x$$

$$= 1 + x^2 - x - \frac{x^3}{6}.$$

$$y^{(2)}(0.2) = 0.83367.$$

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3rd $\quad y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$

$$= 1 + \int_0^x [x - y^{(2)}]\,dx$$

$$= 1 + \int_0^x x - \left[1 + x^2 - x - \frac{x^3}{6}\right] dx$$

$$= \cancel{1} + \int_0^x \left(x - \cancel{x} - x^2 + \cancel{x} + \frac{x^3}{6}\right) dx$$

$$= 1 + \left[x^2 - x^2 - x^3 + x^2 + \frac{x^4}{24}\right]_0^x$$

$$= 1 + \cancel{x^2} - \cancel{x^2} - x^3 + x^2 + \frac{x^4}{24}$$

$$= 1 - x^3 + x^2 + \frac{x^4}{24} \qquad x - x - x^2 + x + \frac{x^3}{6}$$

$$2x - x - x^2 + \frac{x^3}{6}$$

~~$y^{(3)}(0.2) =$~~

$$= 1 + \int_0^x \left[2x - 1 - x^2 + \frac{x^3}{6}\right] dx$$

$$= 1 + [x^2]_0^x - [x]_0^x - \frac{[x^3]_0^x}{3} + \frac{[x^4]_0^x}{24}$$

$$= 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{24}$$

$$y^{(3)}(0.2) = 0.8374$$

Here, $y^{(2)}$ and $y^{(3)}$ are same,

$\therefore\ y^{(3)} = 0.8374$. Ans

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Q. Use Picard's Method to approximate y when $x=0.1$ given that $y=1$ when $x=0$

and $\frac{dy}{dx} = \frac{y-x}{y+x}$ Dec 2014, Dec 2017, NOV-2018

Here, $f(x,y) = \frac{y-x}{y+x}$

$x_0 = 0, \quad y_0 = 1.$

1st. $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

$= y_0 + \int_0^x \frac{y_0 - x}{y_0 + x}\, dx$

$= 1 + \int_0^x \left(\frac{1-x}{1+x}\right) dx$

$= 1 + \int_0^x \left(\left(\frac{2}{1+x}\right) - 1\right) dx$

$= 1 + \left[2\log(1+x) - x\right]_0^x$

$= 1 + (2\log(1+x) - x)$

$y^{(1)}(0.1) = 1.0906.$

2nd. $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$

$= 1 + \int_0^x \frac{(1-x+2\log(1+x))}{(-x+2\log a)\cancel{+x}}\, dx$

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$$y^{(2)} = 1 + \int_0^x \frac{1-2x+2x\log(1+x)}{1+2\log(1+x)}\,dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1+2\log(1+x)}\right] dx$$

$$y^{(2)} = 1 - x - \int_0^x \frac{2x}{1+2\log(1+x)}\,dx$$

This integral can't be solved.
So, first approximation is, $y^{(1)}(0.1) = 1.0906$ Ans.

Q. Use Picard's Method to approximate the value of y at $x = 0.1$ given that $\frac{dy}{dx} = 3x + y^2$ & $y(0) = 1$

$y(0) = 1$. June 2013

Here, $f(x,y) = 3x + y^2$

1st. $y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0)\,dx$

$$= 1 + \int_0^x (3x + y_0^2)\,dx$$

$$= 1 + \int_0^x (3x+1)\,dx$$

$$= 1 + \left[\frac{3x^2}{2} + x\right]_0^x$$

$$= 1 + \frac{3x^2}{2} + x$$

$y^{(1)}(0.1) = 1.115$

2nd. $y^{(2)} = y_0 \int_{x_0}^x f(x, y_1)\,dx$

$$= 1 + \int_0^x \left(3x + \left(1 + x + \frac{3x^2}{2}\right)^2 dx\right.$$

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$= \cancel{1 + \int_0^x 3x + 1 + x^2 + \frac{9x^4}{4} + 2x + 6x^3}$

$y^{(2)} = 1 + \int_0^x \left(\frac{9}{4}x^4 + 3x^3 + 4x^2 + 5x + 1 \right) dx$

$y^{(2)} = \frac{9}{20}x^5 + \frac{3}{4}x^4 - \frac{4}{3}x^3 + \frac{5}{2}x^2 + x + 1$

$y^{(2)}(0.1) = 1.1264$

$y^{(3)} = 1 + \int_{x_0}^x f\left[x, y^{(2)}\right] dx$

$= 1 + \int_0^x \left[3x + \left(\frac{9}{20}x^5 + \frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{5}{2}x^2 + x + 1 \right)^2 \right] dx$

$y^{(3)} = 1 + \int_0^x \Big[\frac{81}{400}x^{10} + \frac{27}{40}x^9 + \frac{141}{80}x^8 + \frac{17}{4}x^7 + \frac{1157}{180}x^6 + \frac{136}{15}x^5 +$

$\frac{125}{12}x^4 + \frac{27}{8}x^3 + 6x^2 + 5x + 1 \Big] dx$

$y^{(3)}(0.1) = 1.12721$

Here, 2^{nd} & 3^{rd} ~~approx~~ approximation are same

So, $y(0.1) = 1.1272$

Q. Perform two iteration of Picard's Method to find an approximate solution of initial value problem.

$\frac{dy}{dx} = x + y^2$ June 2007, Dec 2011

with $y(0) = 1$

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Here, $f(x,y) = x+y^2 \quad y(0) = 1.$

1st. $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

$= 1 + \int_0^x \left[(x + (1)^2 \right] dx$

$= 1 + \frac{x^2}{x} + x$

~~$\Rightarrow y^{(1)}(0.1)$~~ =

$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$

$= 1 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2}\right)^2 dx \right.$

$= 1 + \int_0^x \left[x + 1 + x^2 + \frac{x^4}{4} + 2x + x^3 + x^2 \right] dx$

$= 1 + \frac{x^5}{20} + \frac{x^4}{4} + \frac{2x^3}{3} + \frac{3x^2}{2} + x$ Ans

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2. Taylor's Series Method

Consider the diff eq^n

$$\frac{dy}{dx} = f(x,y).$$

with initial condition

$y = y_0$ at $x = x_0$.

then $\boxed{y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{\underline{|2}} y_0'' + \frac{(x-x_0)^3}{\underline{|3}} y_0''' + \ldots}$

$$y(x) = y_0 + hy_0' + \frac{h^2}{\underline{|2}} y_0'' + \frac{h^3}{\underline{|3}} y_0'' + \ldots$$

where, $h = (x - x_0)$

Q. Use Taylor's Series Method

Solve $\frac{dy}{dx} = x+y$.

given that $y(1) = 0 \rightarrow y_0$, $\rightarrow x_0$.

Find $y(1.2)$ with $h = 0.1$

Here, $y' = x + y$

$x_0 = 0, \quad y_0 = 1$

$y'_0 = x_0 + y_0$

$= 0 + 1 = \underline{1}.$

$\left(\frac{d^2y}{dx^2}\right)$

$(y'') = 1 + y'$

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$(y'')_0 = 1 + (y')_0$

$= 1+1$

$= 2$

$(y''')_0 = 0 + y''$

$= y''$

$(y''')_0 = (y'')_0$

$= 2$

$(y^{iv})_0 = y'''$

$(y^{iv})_0 = (y^{iv})_0 = 2$

$y(x) = y_0 + hy_0' + \frac{h^2}{\lfloor 2} y_0'' + \frac{h^3}{\lfloor 3} y_0''' + \cdots$

$y(1.1) = \cancel{0} + (0.1)(1) + \frac{(0.1)^2}{\lfloor 2}(2) + \frac{(0.1)^3}{\lfloor 3}(2) + \frac{(0.1)^4}{\lfloor 4}(2) + \cdots$

$y(1.1) \cancel{= 1.276}$ 0.11034

Step-2.

$y' = x + y$

$x_1 = 1.1$, $y_1 = 0.11034$.

$(y')_{\cancel{x_1}} = x_1 + y_1$

$= 1.1 + 0.11034$

$(y')_{1.1} = 1.21034$

$(y'') = 1 + y'$

$= 1 + (y')_{x_1}$

$= 1 + (y')_{1.1}$

$(y'')_{1.1} = 1 + 1.21034 = 2.21034.$

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~~$(y''') :=$~~

$$y(1.2) = (0.11034) + (0.1)(1.21034) + \frac{(0.1)^2}{\lfloor 2}(2.2103) + \frac{(0.1)^3}{\lfloor 3}(2.2103) + \dots$$

$$\boxed{y(1.2) = 0.2427}$$

Q. Use Taylor's Series Method

Solve $\frac{dy}{dx} = 2y + 3e^x$ June 2004

given that $y(0) = 0$.

Find y at $x = 0.2$ (x_1) $h =$

$$h = x_1 - x_0$$
$$= 0.2 - 0 = 0.2$$

Here, $y' = 2y + 3e^x$

$x_0 = 0$, $\frac{dy}{dx} = y' = 2y + 3e^x$

$y_0 = 0$

$$(y')_0 = \cancel{x_0 + y_0}\ 2y + 3e^x$$
$$= \cancel{0} = 2(0) + 3e^{(0)} = 3$$

$$(y'') = \cancel{1+0}\ 2(y') + 3e^x$$
$$(y'')_0 = 2(y')_0 + 3e^{x_0}$$
$$= 6 + 3$$
$$= 9$$

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$(y''') = 2(y'') + 3e^{x}$

$(y''')_0 = 2(y'')_0 + 3e^{x_0}$

$= 18 + 3$

$= 21.$

$(y^{iv}) = 2(y''') + 3e^{x}$

$(y^{iv})_0 = 2(y''')_0 + 3e^{x_0}$

$= 2 \times 21 + 3$

$= 42 + 3 = 45$

$(y^{v}) = 2(y^{iv}) + 3e^{x}$

$(y^{v})_0 = 2(y^{iv})_0 + 3e^{0}$

$= 2 \times 45 + 3$

$= 90 + 3 = 93.$

$$y(0.2) = 0 + (0.2)(3) + \frac{(0.2)^2}{\lfloor 2}(9) + \frac{(0.2)^3}{\lfloor 3}(21)$$

$$+ \frac{(0.2)^4}{\lfloor 4} \times (45) + \frac{(0.2)^5}{\lfloor 5} \times 93$$

$$\boxed{y(0.2) = 0.811048.}$$

Q. Use Taylor's Series Method. Solve $\frac{dy}{dx} = x + y^2$.

Given that June 2002

$y(0) = 1$

Find y at $x = 0.1$.

$h = x_1 - x_0$.

$0.1 - 0 = 0.1$

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$y' = x + y^2$

$x_0 = 0$

$y_0 = \cancel{0}.1$

$(y') = \cancel{2y+} \quad x_0 + (y_0)^2$

$(y')_0 = \cancel{2} \quad 1 + 0$

$= 1$

$(y'') = \cancel{x + y^2} \quad 1 + 2yy'$

$\cancel{= x + (y')^2}$ $\quad (y'')_0 = 1 + 2(y_0)(y')_0$

$\cancel{= 1 +}$ $\quad = 1 + 2(1)(1)$

$= 3.$

$\cancel{(y''')_0 = 0 + 2[y' \cdot y'' + y' \cdot y']}$ $\quad (y''')_0 = 2(y)_0(y'')_0 + 2(y')_0^2$

$\cancel{= 0 + 2(1 \cdot 3 + 1 \cdot 1)}$ $\quad \cancel{= 2 \times 0 \times 3 + 2 \times 1}$

$\cancel{= 0 + 2 \times 4}$ $\quad 4$

$\cancel{= 8}$

$\cancel{(y^{iv})_0 =}$

$(y''') = 0 + 2[y \cdot y'' + y' \cdot y']$

$= 2[yy''] + 2[y'^2]$

$= 2(y)_0 (y'')_0 + 2(y')_0^2$

$= 2 \cdot 1 \cdot 3 + 2 \cdot 1^2$

$= 6 + 2 = 8$

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$$(y^{iv}) = 2[y \cdot y''' + (y') \cdot y''] + 4y' \cdot y'']$$

$$= 2[(y)_0 (y''')_0 + (y')_0 (y'')_0] + 4(y')_0 \cdot (y'')_0]$$

$$= 2\cancel{[1 \cdot 8 + 3 \cdot 3 + 4 \cdot 1 \cdot 3]}$$

$$= \cancel{2[8 + 9 + 12]} \qquad = 2[1 \cdot 8 + 1 \cdot 3) + 4 \cdot 1 \cdot 3$$

$$= \cancel{2 \cdot 29} \qquad = 2 \times [8 + 3) + 12$$

$$= \cancel{58} \qquad = 22 + 12 = 34$$

$$y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{\lfloor 2}(3) + \frac{(0.1)^3}{\lfloor 3}(8) + \frac{(0.1)^4}{\lfloor 4}(34)$$

~~y(0.1) = 1.11657~~

$$\boxed{\cancel{y(0.1) = 1.116575}}$$

$$y(0.1) = 1.11647$$

34+12
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Q. Find Taylor Series Method, the value of y at $x = 0.1$ from $\frac{dy}{dx} = x^2y - 1$ given $y(0) = 1$

Dec 2013, June 2014

$$y' = x^2y - 1$$

$$x_0 = 0$$

$$y_0 = 1$$

$$h = x_1 - x_0 = 0.1 - 0 = 0.1.$$

$$(y')_0 = x_0^2 y_0 - 1 = 0 - 1 = -1$$

$$\cancel{= 1 \cdot 0 - 1 = 0}$$

$$= \cancel{x^2 y' + y \cdot 2x - 1}$$

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$(y'') = x^2y' + y \cdot 2x$

$(y'')_0 = \cancel{0}\ 0 \cdot -1 + 1 \cdot 2 \cdot 0$

$= 0.$

$(y''') = [x^2y'' + y'(2x)] + 2[xy' + y(1)]$

$= [0 \cdot 0 + (-1)(2 \cdot 0)] + 2[0 \cdot -1 + (-1)(-1)]$

0 + 1

$= 0 + 2 \times 1$

$= 2.$

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{\lfloor 2}(0) + \frac{(0.1)^3}{\lfloor 3}(2)$$

$$\boxed{y(0.1) = 0.90033.}$$

Euler Modified Method

Consider the difference equation

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0.$

then,

$$y_1 = y^{(1)} = y_0 + h\ f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$y_2 = y^{(2)} = y_1 + h\ f\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$y_n = y^{(n)} = y_{n-1} + h\ f\left[x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{h}{2} f(x_{n-1}, y_{n-1})\right]$$

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Q Use Euler modified method to compute y for $x = 0.005$ given that $\frac{dy}{dx} = x+y$.

June-2014

and $y_0 = 1$, $x_0 = 0$
result correct to three decimal place.

Here $f(x,y) = x+y$
$x_0 = 0$, y_0
and $x_1 = 0.005$

Here $h = x_1 - x_0$
$h = 0.005 - 0$
$\boxed{h = 0.005}$

$$y_1 = y_0 + hf\left[\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]\right.$$

$$= 1 + 0.005 f\left[\left(0 + \frac{0.005}{2}, 1 + \frac{0.005}{2} f(0,1)\right)\right]$$

$$= 1 + 0.005 f\left(0.0025, 1.0025\right)$$

$$= 1 + 0.005 \left(0.0025 + 1.0025\right)$$

$$= 1 + 0.005 \left[1.005\right]$$

$$= 1 + 0.01$$

$\boxed{y_1 = 1.01}$ $y_1 = 1.0525$ (at $h = 0.05$).

$h = 0.05$, $\boxed{y(0.05) = 1.0525}$

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Q. Solve by Euler Modified method to compute y for $x = 1.2$ and 1.4, with $h = 0.2$

$\frac{dy}{dx} = \log(x+y)$ and $y(1) = 2$.

Dec 2004, June 2009, Dec 2017

Here $f(x,y) = \log(x+y)$.

$x_0 = 1$ $y_0 = 2$

$x_1 = 1.2$

$x_2 = 1.4$

$h = x_1 - x_0 = 1.2 - 1 = 0.2$

$h = 0.2$.

$y_1 = y_0 + hf\left[\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$

$= 2 + 0.2f\left[\left(1 + \frac{0.2}{2}, 2 + \frac{0.2}{2} f(1,2)\right)\right]$

$= 2 + 0.2f\left(\left(1 + 0.1, 2 + 0.1\left[\log(1+2)\right]\right]\right.$

$= 2 + 0.2f\left[1.1, 2 + 0.1(1.098)\right]$

$= 2 + 0.2f(1.1, 2 + (0.1098))$

$= 2 + 0.2f(1.1, 2.1098)$

$= 2 + 0.2\left[\log(1.1 + 2.1098)\right]$

$y_1 = 2.2332$.

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$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$= 2.2332 + 0.2f\left[1.2 + \frac{0.2}{2},\ 2.2332 + \frac{0.2}{2} f(1.2, 2.2332)\right]$$

$$= 2.2332 + 0.2f\left[1.3,\ 2.2332 + 0.1\ \underbrace{\left[\log(1.2+2.2332)\right]}_{(=1.2334)}\right]$$

$$= 2.2332 + 0.2f\left[1.3,\ \underbrace{2.2332 + 0.123349}_{[=2.35659]}\right]$$

$$= 2.2332 + 0.2(1.296517)$$

$$y_2 = 2.4925$$

Q. Use Euler Modified method to obtain $y(0.2)$, $y(0.4)$ and $y(0.6)$ correct to three decimal place. June-2013

Given that $\frac{dy}{dx} = y - x^2$ and $y(0) = 1$

$f(x,y) = y - x^2$

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.2$

$x_2 = 0.4$

$x_3 = 0.6$

$h = x_1 - x_0$

$= 0.2 - 0$

$= 0.2$

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$$y_1 = y_0 + hf\left[\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$$

$$= 1 + 0.2 f\left[\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2} f(0,1)\right)\right]$$

$$\cancel{= 1 + 0.2 f\left[0.1, 1 + 0.1 \log(0+1)\right]}$$

$$= 1 + 0.2 f\left[0.1, 1\right]$$

$$y_1 = 1 + 0.2 f\left[\left(0 + \frac{0.2}{2}, 1 + 0.1\right]\right.$$

$1 - 0^2$

$$= 1 + 0.2 f\left[0.1, 1.1\right]$$

$y - x^2$

$1.1 - (0.1)^2$

1.118

$$= 1 + 0.2 (1.09)$$

$$y_1 = 1.218$$

$$y_2 = y_1 + hf\left[\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right)\right]$$

$$= 1.218 + 0.2 f\left[0.2 + \frac{0.2}{2}, 1.218 + \frac{0.2}{2} f(0.2, 1.218)\right]$$

$$= 1.218 + 0.2 f\left[0.3, 1.3358\right]$$

$$= 1.218 + 0.2 (1.2458)$$

$$= 1.46716$$

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$y_3 = y_2 + hf\left[\left(x_2 + \frac{h}{2},\ y_2 + \frac{h}{2} f(x_2, y_2)\right)\right]$

$= 1.467 + 0.2 f\left[0.4 + \frac{0.2}{1},\ 1.467 + \frac{0.2}{2} f(0.4, 1.4671)\right]$

$= 1.467 + 0.2 f\left[0.5,\ 1.5977\right]$

$= 1.73654 \approx 1.737.$

$\boxed{y(0.6) = 1.737}$ Ans

$y(0.4) = 1.46716$

$y(0.2) = 1.218.$

Euler Method

Consider the diff. eq^n

$$\frac{dy}{dx} = f(x, y)$$ $\qquad h = \frac{x - x_0}{n}.$

with $y(x_0) = y_0$

then $\boxed{y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})}$

Q. Use Euler method. Solve in six step.

$$\frac{dy}{dx} = x + y.$$

Dec - 2003

with $y(0) = 0$

Choosing $h = 0.2$

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Here, $f(x,y) = x+y$

$x_0 = 0$

$y_0 = 0.$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_0 + 2h = 0.4$

$x_3 = x_0 + 3h = 0.6$

$x_4 = x_0 + 4h. = 0.8$

$x_5 = x_0 + 5h = 1$

$\cancel{x_6 = x_0 + 6h. = 1.2}$

$y_1 = y_0 + hf(x_0, y_0)$

$= 0 + 0.2 [x_0 + y_0]$

$= 0 + 0.2 [0+0]$

$= 0$

$y_2 = y_1 + hf(x_1, y_1)$

$= 0 + 0.2 [0.2 + 0)$

$= \cancel{0.12} \ 0.04$

$y_3 = y_2 + hf(x_2, y_2)$

$= 0.04 + 0.2 [0.4, 0.04].$

$= \cancel{0.208}. \ 0.128.$

$y_4 = y_3 + hf(x_3, y_3)$

$= 0.128 + 0.2 [0.6 + 0.128)$

$= 0.2736$

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$y_5 = y_4 + hf(x_4, y_4)$
$= 0.2736 + 0.2[0.8 + 0.2736]$
$= 0.48832$

$y_6 = y_5 + hf(x_5, y_5)$
$= 0.48832 + 0.2[1 + 0.48832]$
$= 0.7859.$

Here $y_6 = y(x_6) = y(1.2) = 0.78598.$

~~Q. Use Euler Method Solve in Six step~~

Q. Find $y(0.2)$ Using Euler method from the eq^n
$\frac{dy}{dx} = -xy^2$ with $y(2) = 1$

Taking $h = 0.04$

Here $f(x,y) = -xy^2$
$x_0 = 2$
$y_0 = 1$
$h = 0.04.$

$x_1 = x_0 + h = 2 + 0.04 = 2.04$
$x_2 = x_0 + 2h = \cancel{2.2}\ 2.08$
$x_3 = x_0 + 3h = \cancel{2.2}\ 2.12$
$x_4 = x_0 + 4h = 2.16$
$x_5 = x_0 + 5h = 2.2\ \cancel{2.16}$
$\cancel{x_6 = x_0 + 6h =}$

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$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.04\left[-2(1)^2\right]$$
$$= \cancel{1 + 0.04 \times}$$
$$= 0.92$$

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 0.92 + 0.04\left[-2.04\,(0.92)^2\right]$$
$$= 0.8509$$

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 0.8509 + 0.04\left[-2.08\,(0.8509)^2\right]$$
$$= 0.7906$$

$$y_4 = y_3 + hf(x_3, y_3)$$
$$= 0.7906 + 0.04\left[-2.12\,(0.7906)^2\right]$$
$$= 0.7387$$

$$y_5 = y_4 + hf(x_4, y_4)$$
$$= 0.7387 + 0.04\left[-2.16\,(0.7387)^2\right]$$
$$= 0.6907$$

Hence $y_5 = y(x_5) = y(2.2) = 0.6907$

Q. Apply Euler method solve for y at $x = 0.6$ from $\frac{dy}{dx} = 1 - 2xy$

June-2006

with $y(0) = 0$

take $h = 0.2$

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Here $f(x,y) = 1 - 2xy$

$x_0 = 0$

$y_0 = 0$

$h = 0.2$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_0 + 2h = 0.4$

$x_3 = x_0 + 3h = 0.6.$

$y_1 = y_0 + hf(x_0, y_0)$

$= 0 + 0.2[1 - 2(0)(0)]$

$= 0.2$

$y_2 = y_1 + hf(x_1, y_1)$

$= 0.2 + 0.2[1 - 2(0.2)(0.2)]$

$= 0.384$

$y_3 = y_2 + hf(x_2, y_2)$

$= 0.384 + 0.2[1 - 2(0.4)(0.384)]$

$= 0.52256$

$\cancel{y_4 = y_3 + hf(x_3, y_3)}$

$\cancel{= 0.52256 + 0.2[1 - 2(0.6)(0.52256)]}$

$\cancel{\doteq 0.5971}$

Hence $y_3 = y(x_3) = 0.52256$

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Q. Using Euler method to solve diff. eq^n for y at $x=1$ in five step

$$\frac{dy}{dx} = x^2 + y^2$$ Dec 2008

with $y(0) = 1$

Here $f(x,y) = x^2 + y^2$

$x_0 = 0$

$y_0 = 1.$

take $h = \frac{x - x_0}{5} = \frac{1-0}{5} = 0.2$

$x_1 = x_0 + h = 0.2$

$x_2 = x_0 + 2h = 0.4$

$x_3 = x_0 + 3h = 0.6.$

$x_4 = x_0 + 4h = 0.8$

$x_5 = x_0 + 5h = 1.0.$

$y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.2 (0^2 + 1^2)$

$= \cancel{0.2}.\ 1.2$

$y_2 = y_1 + hf(x_1, y_1)$

$= 1.2 + 0.2 ((0.2)^2 + (1.2)^2)$

$= 1.496$

$y_3 = y_2 + hf(x_2, y_2)$

$= 1.496 + 0.2 ((0.4)^2 + (1.496)^2)$

$= 1.9756$

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$y_4 = y_3 + hf(x_3, y_3)$

$= 1.9756 + 0.2\left((0.6)^2 + (1.9756)^2\right)$

$= 2.8281.$

$y_5 = y_4 + hf(x_4, y_4)$

$= 2.8281 + 0.2\left(0.8^2 + 2.8281^2\right)$

$= 4.5557$

Hence $y_5 = y(x_5) = y(1) = 4.5557.$

Q. Using Euler Method to solve diff eq^n at $x = 0.1$

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$y(0) = 1$ take $h = 0.02.$

Here $f(x, y) = \dfrac{y-x}{y+x}$

$x_0 = 0$, $y_0 = 1.$

$x_1 = x_0 + h = 0.02$

$x_2 = x_0 + 2h = 0.04$

$x_3 = x_0 + 3h = 0.06$

$x_4 = x_0 + 4h = 0.08$

$x_5 = x_0 + 5h = 0.1.$

$y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.02\left[\dfrac{1-0}{1+0}\right]$

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$= 1.02.$

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$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.02 + 0.02\left[\frac{1.02 - 0.02}{1.02 + 0.02}\right]$$
$$= 1.0392$$

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 1.0392 + 0.02\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$
$$= 1.0577$$

$$y_4 = y_3 + hf(x_3, y_3)$$
$$= 1.0756$$

$$y_5 = y_4 + hf(x_4, y_4)$$
$$= \cancel{1.0756 + 0.02}$$
$$= 1.0928$$

Hence, $y_5 = y(x_5) = y(0.1) = 1.0928$.

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* Runge - Kutta Method
OR
Runge - Kutta Fourth order.

Consider the diff. eq^n

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0$

& calculate

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2, \; y_0 + K_1/2)$$

$$K_3 = hf(x_0 + h/2, \; y_0 + K_2/2)$$

$$K_4 = hf(x_0 + h, \; y_0 + K_3)$$

Compute

$$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

Req. value of y.

$$\boxed{y = y_0 + K}$$

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Q. Apply Runge - Kutta method to find an approximate value of y at $x = 0.2$.

Given that

$$\frac{dy}{dx} = x+y \quad \text{June 2010, Dec 2013}$$

with $y(0) = 1$.

$x_0 = 0$

$y_0 = 1$

$$\frac{dy}{dx} = x+y$$

$h = x_1 - x_0 = 0.2 - 0 = 0.2$

$K_1 = hf(x_0, y_0) = 0.2[x_0 + y_0] = 0.2.$

$K_2 = hf\left(x_0 + \frac{h}{2} + y_0 + \frac{K_1}{2}\right)$

$= 0.2\left[0 + \frac{0.2}{2} + 1 + \frac{0.2}{2}\right]$

$= 0.24$

$K_3 = hf\left[x_0 + \frac{h}{2} + y_0 + \frac{K_2}{2}\right]$

$= 0.2\left[0 + \frac{0.2}{2} + 1 + \frac{0.24}{2}\right]$

$= 0.244$

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$K_4 = hf\left[x_0 + h, y_0 + K_3\right]$

$= 0.2\left[0 + \frac{0.2}{2} + 1.244\right]$

$= 0.2888.$

Now, $K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.2 + 2\times 0.24 + 2\times 0.244 + 0.2888\right]$

$K = 0.2428.$

Req. values of y.

$y = y_0 + K$

$= 1 + 0.2428$

$= 1.2428.$

$\boxed{y(0.2) = 1.2428}$

Q Apply Runge Method to find an approximate value of y for $x = 0.2$ in steps of 0.1, ~~if~~ $\cancel{\frac{dy}{dx} = x+y^2}$ **Dec-2004, Feb.2010, June-2011, June-2012, Dec 2012**

Given that $\frac{dy}{dx} = x + y^2$

with $y(0) = 1$.

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$$\frac{dy}{dx} = x+y^2$$

$h = 0.1$

$x_0 = 0$

$y_0 = 1.$

Step 1. $K_1 = hf(x_0, y_0) = 0.1\left[0+1^2\right] = 0.1.$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.1\left[\frac{0.1}{2} + \left(1+\frac{0.1}{2}\right)^2\right].$

$= 0.11525$

$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= 0.1\left[0+\frac{0.1}{2} + \left(1+\frac{0.11525}{2}\right)^2\right]$

$= 0.116857$

$K_4 = hf\left[x_0+h, y_0+K_3\right]$

$= 0.1\left[0.1 + \left(1+0.116857\right)^2\right]$

$= 0.13473695$

$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= 0.116491825$

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Req. value of y.

$$y = y_0 + K$$
$$= 1 + 0.116491825$$
$$= 1.116491825$$

$$y(0.1) = 1.116491825 \approx 1.1165$$

Step-2 Now, $x_1 = 0.1$
$y_1 = 1.116491825 \approx 1.1165$

$$K_1 = hf(x_1, y_1) = 0.1\left[0.1 + (1.1165\cancel{913})^2\right]$$

$$= 0.134657$$

$$K_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right]$$

$$= 0.15514416$$

$$K_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right]$$

$$= 0.15284452 \approx 0.1576$$

$$K_4 = hf\left[x_1 + h, y_1 + K_3\right]$$

$$= 0.1811224 \approx 0.1823$$

$$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= 0.1552912 \approx 0.1571$$

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Req. value of y.

$= y = y_1 + K$

$= 1.1165 + 0.15529$

$= 1.27179. \approx 1.2736$

$\underline{y(0.2) = 1.27179. \approx 1.2736.}$

Q. Use Runge Kutta method of fourth order, solve the diff. eq^n June-2003

$\frac{dy}{dx} = xy.$ for $x = 1.2.$

Given that $y(1) = 2$ (taking $h = 0.1$)

$x_0 = 1$

$y_0 = 2$

$h = 0.1$

$\frac{dy}{dx} = xy.$

$K_1 = hf[x_0, y_0] = 0.1[1 \times 2] = \cancel{0.12}\ 0.2$

$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right]$

$= 0.2205$

$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= 0.221576$

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$K_4 = hf[x_0 + h, y_0 + K_3)$
$= 0.24437$

$K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 0.22142$

Req. value of y

$y = y_0 + K$
$= 2 + 0.22142$
$= 2.22142$

~~$x_1(1.1)$~~ =
$y(1.1) = 2.2214$

$x_1 = 1.1$
$y_1 = 2.2214.$

$K_1 = hf[x_0, y_0] = \cancel{0.1}[\quad 0.24435$

$K_2 = hf[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}].$
$= 0.1[(1.1 + \frac{0.1}{2}) \times (2.2214 + \frac{0.24435}{2})]$

$= 0.2676.$
$K_3 = hf[x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2})$

$= 0.27084.$

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$K_4 = hf\left(x_0+h,\ y_0+K_3\right)$

$= 0.1\left[\left((1.1+0.1)\times(2.22142+0.2708)\right)\right]$

$= 0.299066$

$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= 0.27005$

$y_2 = y_1 + K$

$= 1.1 + 0.27005$

$= 1.37005$

$y(0.2) = 1.37005$.

Q. Use Runge Kutta method of fourth order. Solve the diff. eq^n. $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$. at $x = 0.2$ and $x = 0.4$

June 2004, June-2008, June-2016

Step-1 $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

$x_0 = 0.$

$y_0 = 1$

$h = x_1 - x_0 = 0.2 - 0$

$h = 0.2$

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$K_1 = hf[x_0, y_0]$

$= 0.2\left[\frac{1^2 - 0^2}{1^2 + 0^2}\right]$

$= 0.2$

$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right]$

$= 0.2f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$

$= 0.2f[0.1, 1.1]$

$= 0.19672$

$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= 0.2[0.1, 1.09836]$

$= 0.1967$

$K_4 = hf[x_0 + h, y_0 + K_3]$

$= 0.2f[0.2, 1 + 0.1967]$

$= 0.2f[0.2, 1.1967]$

$= \cancel{1.18913}$ 0.1891

$K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 0.19599$

$y = y_0 + K$

$= 1 + 0.19599 = 1.19599$

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$y(0.2) = 1.196$

Step-2:

$x_1 = 0.2$

$y_1 = 1.196$

$h = 0.2$

$k_1 = hf[x_1, y_1]$

$= \cancel{0.2}\ 0.189118$

$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right]$

$= 0.2\left[0.2 + \frac{0.2}{2}, 1.196 + \frac{0.189}{2}\right]$.

$= 0.2[0.3, 1.2905]$

$= 0.17949$

$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right]$

$= 0.2[0.3, 1.285745]$

$= 0.17934$

0.2
0.2
1.196
0.17934
1.37534

$k_4 = hf[x_1 + h, y_1 + k_3]$

$= 0.2(0.4, 1.37534)$

$= 0.1688$

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$$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= 0.17925$$

$$y(0.4) = y_1 + K$$
$$= 1.196 + 0.1792$$
$$= 1.3752.$$

$$y(0.4) = \underline{\underline{1.3752.}}$$

* <u>Milne's Predictor - Corrector Formula</u> :-

Consider the diff. eq^n

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0$
and y_1, y_2, y_3 are also given

then, $$\boxed{y_{4p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]}$$

which is known as <u>Milne Predictor formula</u>
and

$$\boxed{y_{4c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]}$$

which is known as <u>Miln's Corrector formula</u>

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Q. Given $\frac{dy}{dx} = x^2(1+y)$

and $y(1) = 1.$

$y(1.1) = 1.233.$
$y(1.2) = 1.548,$
$y(1.3) = 1.979.$

Evaluate $y(1.4)$ by Milne Predictor corrector formula.

Given,

$y' = f(x,y) = x^2(1+y)$

$x_0 = 1, \quad x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3.$
$y_0 = 1, \quad y_1 = 1.233, \quad y_2 = 1.548, \quad y_3 = 1.979$
$h = 0.1$

$\cancel{y_0' = f(x_0, y_0) = x^2(1+y) = 2}$
$y_1' = f(x_1, y_1) = x^2(1+y) = 2.70193$
$y_2' = f(x_2, y_2) = 3.669.$
$y_3' = f(x_3, y_3) = 5.03451.$

y_4 Using Milne's Predictor formula,

$$y_{4p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right].$$

$$= 1 + \frac{4\times 0.1}{3}\left[2\times 2.7019 - 3.669 + 2\times 5.03451\right]$$

$$= 2.57384$$

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Now, $y_4' = f(x_4, y_4)$

$= x_4^2 (1 + y_4)$

$= (1.4)^2 [1 + 2.57384]$

$= 7.0047.$

Using Milne's Corrector formula.

$y_{4c} = y_2 + \frac{h}{3} [y_2' + 4y_3' + y_4']$

$= 1.548 + \frac{0.1}{3} [3.669 + 4 \times 5.03451 + 7.0047]$

$y_{4c} = 2.5750$

Then $y(x_4) = y(1.4) = 2.5750.$

Q. The diff. eq^n $\frac{dy}{dx} = 1 + y^2$ satisfies the following sets of values of x and y.

x	0	0.2	0.4	0.6
y	0	0.2027	0.4228	0.6841

Compute $y(0.3)$ using Milne's Predictor formula

$y' = f(x, y) = 1 + y^2$

$x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

$y_0 = 0$, $y_1 = 0.2027$, $y_2 = 0.4228$, $y_3 = 0.6841$

$h = 0.2$

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$y_1' = f(x_1, y_1) = 1+y^2 = 1.04108.$
$y_2' = f(x_2, y_2) = 1.178759$
$y_3' = f(x_3, y_3) = 1.46799$

Using Milne's Predictor formula

$$y_{4p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1.0238$$

Now, $y_4' = f(x_4, y_4)$
$= 1+y_4^2$
$= 1+1.0238^2$
$= 2.0481$

Using Milne's Corrector ~~Method~~ formula -

$$y_{4c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 0.4228 + \frac{0.2}{3}\left[1.178759 + 4 \times 1.46799 + 2.0481\right]$$

$$= \cancel{1.439} \quad 1.02938.$$

Q. Apply Milne's Predictor - Corrector formula to find a solution of diff. eq^n.

$$\frac{dy}{dx} = x-y^2$$

Dec 2005, Dec 2013

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in the range $0 \leq x \leq 1$ for bonding condition $y = 9$ at $x = 0$.

First we compute y_1, y_2, y_3 by using Picard's method.

Here $y' = f(x,y) = x - y^2$

$x_0 = 0$ $y_0 = 0$. taking $h = 0.2$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_0 + 2h = 0.4$.

$x_3 = x_0 + 3h = 0.6$.

$x_4 = x_0 + 4h = 0.8$.

$x_5 = x_0 + 5h = 1.0$.

Using Picard's Method.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 0 + \int_0^x [x - y_0^2].$$

$$= \int_0^x x\, dx = \frac{x^2}{2}$$

$$y^2 = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$= 0 + \int_0^x \left[\left(x - \left(\frac{x^2}{2}\right)^2\right) dx.\right.$$

$$y^2 = \frac{x^2}{2} - \frac{x^5}{20}$$

$\therefore$ Hence $y = \frac{x^2}{2} - \frac{x^5}{20}$

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$x_0 = 0$ $y_0 = 0$

$x_1 = 0.2$ $y_1 = 0.02$

$x_2 = 0.4$ $y_2 = 0.0795$

$x_3 = 0.6$ $y_3 = 0.176$

~~Using Milne's Predictor formula~~

$$\cancel{y_{4p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]}.$$

$$y_1' = f(x_1, y_1) = x - y^2 = 0.1996.$$

$$y_2' = 0.3937$$

$$y_3' = 0.5690$$

Using Milne's Predictor formula

$$y_{4p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0.30493$$

Now, $y_4' = f(x_4, y_4)$

$$= x - y^2$$

$$= 0.8 - (0.3049)^2$$

$$= 0.707.$$

Using Milne's Corrector Method formula

$$y_{4c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 0.30461$$

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Hence, $y(0.8) = 0.3046.$

Again,

Using Milne's Predector formula.

$$y_{5p} = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$= 0.1996 + \frac{4 \times 0.2}{3}\left[2 \times 0.3937 - 0.5690 + 2 \times 0.707\right]$$

$$= 0.4554$$

$$y_5' = f(x_5, y_5)$$
$$= (1.0)^2 - (0.4554)^2 = 0.7926.$$

Using Milne's Corrector formula

$$y_5 = y_3 + \frac{h}{3}\left[y_3' + 4y_4' + y_5'\right]$$

$$= 0.4553$$

Hence, $y = 0.4554$ at $x = 1.0.$

$y(1.0) = 0.4554.$

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Q. Use Milne's formula to find $y(0.3)$ from $\frac{dy}{dx} = x^2+y^2$, $y(0) = 1$ — June 2006, June 2007

Find initial value $y(-0.1)$, $y(0.1)$, $y(0.2)$ from the Taylor series method.

Given $\frac{dy}{dx} = y' = x^2+y^2$

$x_0 = 0$, $y_0 = 1$.

$$y' = x^2 + y^2$$
$$y'' = 2x + 2yy'$$
$$y''' = 2 + 2[y \cdot y'' + y' \cdot y'] = 2 + 2yy'' + 2[y']^2$$
$$y^{iv} = 2[y \cdot y''' + y'' \cdot y'] + 4y' \cdot y''$$
$$= 2y''' + 6y'y''$$

Here, $x_0 = 0$, $y_0 = 1$.

$$y_0' = x_0^2 + y_0^2 = 0 + 1 = 1.$$
$$y_0'' = 2x_0 + 2y_0y_0' = 2$$
$$y_0''' = 2 + 2[y_0 \cdot y_0'' + y_0' \cdot y_0'] =$$
$$= 2 + 2y_0y_0'' + 2[y_0']^2 = 8$$
$$y^{iv} = 2y_0''' + 6y_0'y_0'' = 28$$

From Taylor's Series.

$$y(x) = y_0 + hy_0' + \frac{h^2}{\lfloor 2} y_0'' + \frac{h^3}{\lfloor 3} y_0''' + \frac{h^4}{\lfloor 4} y_0^{iv} + \ldots$$

$$y(x) = 1 + 0.1 \times 1 + \frac{(0.1)^2}{\lfloor 2} \times 2 + \frac{(0.1)^3}{\lfloor 3} \times 8 + \frac{(0.1)^4}{\lfloor 4} \times 28$$

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From Taylor's formula.

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{\lfloor 2}y_0'' + \frac{(x-x_0)^3}{\lfloor 3}y_0''' + \frac{(x-x_0)^4}{24}y_0^{iv}$$

$$y(x) = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{6} \times 8 + \frac{x^4}{24} \times 28$$

$y(-0.1) = 0.9087$
$y(0.1) = 1.11145$
$y(0.2) = 1.25253$

Hence,

x	$x_0 = 0$	$x_1 = -0.1$	$x_2 = 0.1$	$x_3 = 0.2$
y	$y_0 = 1$	$y_1 = 0.9087$	$y_2 = 1.11145$	$y_3 = 1.25253$.
y'	$y_0' = 1$	$y_1' = 0.8357$	$y_2' = 1.2453$	$y_3' = 1.6088$

Now, we shall find the value y at $x = 0.3$

Using Milne's Predictor formula

$$y_4 = y_0 + \frac{4h}{3}\left[2y' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4 \times 0.1}{3}\left[2 \times 0.8357 - 1.2453 + 2 \times 1.6088\right]$$

$$= 1.48582$$

Now $y_4' = f(x_4, y_4)$

$= \cancel{x_4^2(1+y_4)}\ x_4^2 + y_4^2$

$= (0.3)^2 + (1.48582)^2 = 2.29766$

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Now, $y_{4C} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$= 1.11145 + \frac{0.1}{3}[1.2453 + 4 \times 1.6088 + 2.29766]$

$y(0.3) = 1.44405.$

Adams Bashforth Method -

$(y_{n+1})_p = y_n + \frac{h}{24}[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$

which is know as Predictor formula and corrector formula.

$(y_{n+1})_c = y_n + \frac{h}{24}[9y_{n+1}' - 19y_n' - 5y_{n-1}' + y_{n-2}']$

This is called Adams Basforthe corrector formul.

Ques. — Given $\frac{dy}{dx} = x^2(1+y)$ and

$y(1) = 1$, $y(1.1) = 1.233$

$y(1.2) = 1.548$ $y(1.3) = 1.979$

evaluate $y(1.4)$ by

Adams Bashforth method.

Sol Here $f(x,y) = y' = x^2(1+y)$

n	x	y	$y' = x^2(1+y)$	
0	1.0	1.000	$1(1+1) = 2$	y_0'
1	1.1	1.233	$(1.1)^2(1+1.233) = 2.702$	y_1'
2	1.2	1.548	$(1.2)^2(1+1.548) = 3.66912$	y_2'
3	1.3	1.979	$(1.3)^2[1+1.979] = 5.035$	y_3'

obviously the interval of differentia

$h = 0.5$

Adams ~~Milne~~ Bashforms Pridictor formula.

$$y_4 = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$= 1.979 + \frac{0.5}{24}[55(5.035) - 59(3.66912) + 37(2.702) - 9(2)]$$

$= 1.979 + 0.0208[$

$= \underline{\underline{2.573}}$

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$x = 0.4$
$y = 2.573$ $\quad$ Now $y_4' = 7.004$

Corrector formula:

$$(y_4)_c = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[9 \times 7.004 + 19(5.035) - 5(3.66912) + 2.702\right]$$

$$= 2.575$$

$$y(1.4) = 2.575$$

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